Sample Exam
Linear Algebra

1. Pair the following list of vectors down to a basis of $\mathbb{R}^3$.
\[
\begin{bmatrix}
-2 \\
-1 \\
-2
\end{bmatrix},
\begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix},
\begin{bmatrix}
2 \\
1 \\
1
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
3 \\
3 \\
2
\end{bmatrix}
\]

2. Find the dimension of the subspace in $\mathbb{P}_3$ spanned by $\{x^2, x^2-x-1, x+1\}$.

3. Let $A = \begin{bmatrix}
1 & 2 & 2 & 0 \\
0 & 0 & 1 & -1 \\
1 & 2 & 3 & -1
\end{bmatrix}$. Find a basis for the null space $N(A)$, the Row space $\text{row}(A)$ and the column space $R(A) = \text{col}(A)$.

4. Find the angle between the two vectors $(1, 2, 3)$ and $(1, 2, -3)$.

5. Find the vector projection of $(1, -1, 2, 0)$ onto the direction of $(1, 2, 1, 1)$.

6. Which of the following are not linear transformations? Explain why not.
   a) The map from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by $f([x, y, z]^T) = [x^2, yz]^T$.
   b) The map from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by $f([x, y, z]^T) = [2x + y, z, 0]^T$.
   c) The map from $\mathbb{C}([0, 1])$ to $\mathbb{R}^3$ given by $L(f) = \int_0^1 f(x)dx + f(0)$.
   d) The map from $\mathbb{C}([0, 1])$ to $\mathbb{R}$ given by $L(f) = \int_0^1 f(x)dx$.
   e) The map from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}^2 \times \mathbb{R}^2$ given by $F(A) = I + A$.
   f) The map from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}^2 \times \mathbb{R}^2$ given by $L(A) = BA$ where $B = \begin{bmatrix} 1 & -1 \\ 2 & -1/2 \end{bmatrix}$.
   g) The map from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}^2 \times \mathbb{R}^2$ given by $L(A) = A + A^T$.

7. Determine the kernel and range of the map $P_3$ to $P_3$ given by $L(p)(x) = xp'(0)$.

8. Determine the kernel and range of the map $R^2 \times \mathbb{R}^2$ to $\mathbb{R}$ given by $F(A) = \frac{A + A^T}{2}$.

9. Determine a basis for $R(A)$, $R(A^T)$, $N(A)$ and $N(A^T)$ where
\[
A = \begin{bmatrix}
1 & 2 & 1 \\
0 & 1 & 1 \\
1 & 4 & 3
\end{bmatrix}
\]

10. Find the matrix that represents the map $D : P_3$ to $P_3$ given by $D(p) = p'$ with respect to the basis $S = \{1, x, x^2\}$ and with respect to the basis $B = \{1, 2x, 4x^2 - 2\}$ in other words, find $[D]^S_S$ and $[D]^B_B$. Find the transition matrix $S = [id]^S_S$ from the basis $B$ to $S$.
   Show that $[D]^B_B = ([id]^S_S)^{-1} [D]^S_S [id]^S_S$. 