Manifolds and Differential Geometry: Vol II

Jeffrey M. Lee
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0.1 Preface
0.1. PREFACE

Hi
Chapter 1  
Comparison Theorems  

1.1 Rauch’s Comparison Theorem

In this section we deal strictly with Riemannian manifolds. Recall that for a (semi-) Riemannian manifold $M$, the sectional curvature $K_M(P)$ of a 2–plane $P \subset T_p M$ is

$$\langle \mathfrak{R}(e_1 \wedge e_2), e_1 \wedge e_2 \rangle$$

for any orthonormal pair $e_1, e_2$ that span $P$.

**Lemma 1.1** If $Y$ is a vector field along a curve $\alpha : [a, b] \to M$ then if $Y^{(k)}$ is the parallel transport of $\nabla^k_\partial t Y(a)$ along $\alpha$ we have the following Taylor expansion:

$$Y(t) = \sum_{k=0}^m \frac{Y^{(k)}(t)}{k!} (t-a)^k + O(|t-a|^{m+1})$$

**Proof.** Exercise. ■

**Definition 1.2** If $M, g$ and $N, h$ are Riemannian manifolds and $\gamma^M : [a, b] \to M$ and $\gamma^N : [a, b] \to N$ are unit speed geodesics defined on the same interval $[a, b]$ then we say that $K^M \geq K^N$ along the pair $(\gamma^M, \gamma^N)$ if $K^M(Q_{\gamma^M(t)}) \geq K^N(P_{\gamma^N(t)})$ for every $t \in [a, b]$ and every pair of 2–planes $Q_{\gamma^M(t)} \in T_{\gamma^M(t)}M$, $P_{\gamma^N(t)} \in T_{\gamma^N(t)}N$.

We develop some notation to be used in the proof of Rauch’s theorem. Let $M$ be a given Riemannian manifold. If $Y$ is a vector field along a unit speed geodesic $\gamma^M$ such that $Y(a) = 0$ then let

$$I^M_s(Y, Y) := I_{\gamma^M|_{[a,s]}}(Y,Y) = \int_a^s \langle \nabla_{\partial_s} Y(t), \nabla_{\partial_s} Y(t) \rangle + \langle R_{\gamma^M} Y, \gamma^M \gamma^M Y \rangle(t) dt.$$ 

If $Y$ is an orthogonal Jacobi field then $I^M_s(Y, Y) = \langle \nabla_{\partial_s} Y, Y \rangle(s)$ by theorem ??.
Theorem 1.3 (Rauch) Let $M, g$ and $N, h$ be Riemannian manifolds of the same dimension and let $\gamma^M : [a, b] \to M$ and $\gamma^N : [a, b] \to N$ unit speed geodesics defined on the same interval $[a, b]$. Let $J^M$ and $J^N$ be Jacobi fields along $\gamma^M$ and $\gamma^N$ respectively and orthogonal to their respective curves. Suppose that the following four conditions hold:

(i) $J^M(a) = J^N(a)$ and neither of $J^M(t)$ or $J^N(t)$ is zero for $t \in (a, b)$
(ii) $\left\| \nabla_{\dot{\gamma}} J^M(a) \right\| = \left\| \nabla_{\dot{\gamma}} J^N(a) \right\|
(iii) $L(\gamma^M) = \text{dist}(\gamma^M(a), \gamma^M(b))$
(iv) $K^M \geq K^N$ along the pair $(\gamma^M, \gamma^N)$

Then $\left\| J^M(t) \right\| \leq \left\| J^N(t) \right\|$ for all $t \in [a, b]$.

Proof. Let $f_M$ be defined by $f_M(s) := \left\| J^M(s) \right\|$ and $h_M$ by $h_M(s) := f_M(s)^2 / \left\| J^M(s) \right\|^2$ for $s \in (a, b)$. Define $f_N$ and $h_N$ analogously. We have

$$f_M'(s) = 2J^M(s)J^M(s)$$
and the analogous equality for $f_N$ and $h_N$. If $c \in (a, b)$ then

$$\ln(\left\| J^M(s) \right\|^2) = \ln(\left\| J^M(c) \right\|^2) + 2 \int_c^s h_M(s')ds'$$
with the analogous equation for $N$. Thus

$$\ln\left( \frac{\left\| J^M(s) \right\|^2}{\left\| J^N(s) \right\|^2} \right) = \ln\left( \frac{\left\| J^M(c) \right\|^2}{\left\| J^N(c) \right\|^2} \right) + 2 \int_c^s [h_M(s') - h_N(s')]ds'$$

From the assumptions (i) and (ii) and the Taylor expansions for $J^M$ and $J^N$ we have

$$\lim_{c \to a} \frac{\left\| J^M(c) \right\|^2}{\left\| J^N(c) \right\|^2} = 0$$
and so

$$\ln\left( \frac{\left\| J^M(s) \right\|^2}{\left\| J^N(s) \right\|^2} \right) = 2 \lim_{c \to a} \int_c^s [h_M(s') - h_N(s')]ds'$$

If we can show that $h_M(s) - h_N(s) \leq 0$ for $s \in (a, b)$ then the result will follow. So fix $s_0 \in (a, b)$ let $Z^M(s) := J^M(s)/\left\| J^M(s) \right\|$ and $Z^N(s) := J^N(s)/\left\| J^N(s) \right\|$. We now define a parameterized families of sub-tangent spaces along $\gamma^M$ by $W_M(s) := \hat{\gamma}^M(s) \perp T_{\gamma^M(s)}M$ and similarly for $W_N(s)$. We can choose a linear isometry $L_r : W_N(r) \to W_M(r)$ such that $L_r(J^N(r)) = J^M(r)$. We now want to extend $L_r$ to a family of linear isometries $L_s : W_N(s) \to W_M(s)$. We do this using parallel transport by

$$L_s := P(\gamma^M)_r^s \circ L_r \circ P(\gamma^N)_r^s.$$
1.1. RAUCH’S COMPARISON THEOREM

Define a vector field $Y$ along $\gamma^M$ by $Y(s) := L_s(J^M(s))$. Check that

$$
Y(a) = J^M(a) \\
Y(r) = J^M(r) \\
\|Y\|^2 = \|J^N\|^2 \\
\|\nabla_{\partial_t} Y\|^2 = \|\nabla_{\partial_t} J^N\|^2
$$

The last equality is a result of exercise ?? where in the notation of that exercise $\beta(t) := P(\gamma^M) \circ Y(t)$. Since (iii) holds there can be no conjugates along $\gamma^M$ up to $r$ and so $I^M_r$ is positive definite. Now $Y - J^M$ is orthogonal to the geodesic $\gamma^M$ and so by corollary ??we have $I^M_r(J^M, J^M) \leq I^M_r(Y, Y)$ and in fact

$$
I^M_r(J^M, J^M) \leq I^M_r(Y, Y) = \int_a^r \|\nabla_{\partial_t} Y\|^2 + R^M(\dot{\gamma}^M, Y, \dot{\gamma}^M, Y) \\
\leq \int_a^r \|\nabla_{\partial_t} Y\|^2 + R^N(\dot{\gamma}^N, J^N, \dot{\gamma}^N, J^N) \text{ (by (iv))} \\
= I^N_r(J^N, J^N)
$$

Recalling the definition of $Y$ we obtain

$$
I^M_r(J^M, J^M)/\|J^M(r)\|^2 \leq I^N_r(J^N, J^N)/\|J^N(r)\|^2
$$

and so $h_M(r) - h_N(r) \leq 0$ but $r$ was arbitrary and so we are done.

1.1.1 Bishop’s Volume Comparison Theorem

under construction

1.1.2 Comparison Theorems in semi-Riemannian manifolds

under construction
Chapter 2

Submanifolds in Semi-Riemannian Spaces

2.1 Definitions

Let $M$ be a $d$ dimensional submanifold of a semi-Riemannian manifold $\overline{M}$ of dimension $n$ where $d < n$. The metric $g(.,.) = \langle .,. \rangle$ on $\overline{M}$ restricts to tensor on $M$ which we denote by $h$. Since $h$ is a restriction of $g$ we shall also use the notation $\langle .,. \rangle$ for $h$. If the restriction $h$ of is nondegenerate on each space $T_pM$ then $h$ is a metric tensor on $M$ and we say that $M$ is a semi-Riemannian submanifold of $\overline{M}$. If $\overline{M}$ is Riemannian then this nondegeneracy condition is automatic and the metric $h$ is automatically Riemannian. More generally, if $\phi : M \rightarrow \overline{M}, g$ is an immersion we can consider the pull-back tensor $\phi^*g$ defined by

$$\phi^*g(X,Y) = g(T\phi \cdot X, T\phi \cdot Y).$$

If $\phi^*g$ is nondegenerate on each tangent space then it is a metric on $M$ called the pull-back metric and we call $\phi$ a semi-Riemannian immersion. If $M$ is already endowed with a metric $g_M$ then if $\phi^*g = g_M$ then we say that $\phi : (M,g_M) \rightarrow (\overline{M},g)$ is an isometric immersion. Of course, if $\phi^*g$ is a metric at all, as it always is if $(\overline{M},g)$ is Riemannian, then the map $\phi : (M,\phi^*g) \rightarrow (\overline{M},g)$ is an isometric immersion. Since every immersion restricts locally to an embedding we may, for many purposes, assume that $M$ is a submanifold and that $\phi$ is just the inclusion map.

**Definition 2.1** Let $(\overline{M},g)$ be a Lorentz manifold. A submanifold $M \subset \overline{M}$ is said to be spacelike (resp. timelike, lightlike) if $T_pM \subset T_p\overline{M}$ is spacelike (resp. timelike, lightlike).

There is an obvious bundle on $M$ which is the restriction of $T\overline{M}$ to $M$. This is the bundle $T\overline{M}|_M = \bigcup_{p \in M} T_p\overline{M}$. Each tangent space $T_p\overline{M}$ decomposes as

$$T_p\overline{M} = T_pM \oplus (T_pM)\perp$$

7
where \((T_pM)^\perp = \{ v \in T_pM : \langle v, w \rangle = 0 \text{ for all } w \in T_pM \}\). Then \(TM^\perp = \bigsqcup_p (T_pM)^\perp\) its natural structure as a smooth vector bundle called the **normal bundle** to \(M\) in \(\overline{M}\). The smooth sections of the normal bundle will be denoted by \(\Gamma(TM^\perp)\) or \(\mathfrak{X}(M)^\perp\). Now the orthogonal decomposition above is globalizes as

\[
T\overline{M}|_M = TM \oplus TM^\perp
\]

A vector field on \(M\) is always the restriction of some (not unique) vector field on a neighborhood of \(\overline{M}\). The same is true of any not necessarily tangent vector field along \(M\). The set of all vector fields along \(M\) will be denoted by \(X(M)|_M\). If \(X \in X(M)\) then we denote its restriction to \(M\) by \(X |_M\) or sometimes just \(X\).

Since any function on \(M\) is also the restriction of some function on \(\overline{M}\) we may consider \(X(M)|_M\) as a submodule of \(X(M)|_M\). If \(X \in X(M)\) then we denote its restriction to \(M\) by \(X |_M\) or sometimes just \(X\). Notice that \(X(M)^\perp\) is a submodule of \(X(M)|_M\). We have two projection maps: \(T_p\overline{M} \to N_pM\) and \(\text{tan} : T_p\overline{M} \to T_pM\) which in turn give module projections \(\text{nor} : X(M)|_M \to X(M)^\perp\) and \(\text{tan} : X(M)|_M \to X(M)\). The reader should contemplate the following diagrams:

\[
\begin{array}{ccc}
C^\infty(\overline{M}) \xrightarrow{\text{restr}} C^\infty(M) & \times \downarrow \times \downarrow & C^\infty(M) \\
X(\overline{M}) \xrightarrow{\text{restr}} X(\overline{M})|_M \xrightarrow{\text{tan}} X(M) \\
\end{array}
\]

and

\[
\begin{array}{ccc}
C^\infty(\overline{M}) \xrightarrow{\text{restr}} C^\infty(M) & \times \downarrow \times \downarrow & C^\infty(M) \\
X(\overline{M}) \xrightarrow{\text{restr}} X(\overline{M})|_M \xrightarrow{\text{nor}} X(M)^\perp \\
\end{array}
\]

Now we also have an exact sequence of modules

\[
0 \to X(M)^\perp \to X(\overline{M})|_M \xrightarrow{\text{tan}} X(M) \to 0
\]

which is in fact a split exact sequence since we also have

\[
0 \xleftarrow{\text{nor}} X(M)^\perp \xleftarrow{\text{restr}} X(\overline{M})|_M \xleftarrow{\text{tan}} X(M) \xleftarrow{\text{nor}} 0
\]

The extension map \(X(\overline{M})|_M \xleftarrow{\text{nor}} X(M)^\perp\) is not canonical but in the presence of a connection it is almost so: If \(U_\epsilon(M)\) is the open tubular neighborhood of \(M\) given, for sufficiently small \(\epsilon\) by

\[
U_\epsilon(M) = \{ p \in \overline{M} : \text{dist}(p, M) < \epsilon \}
\]

then we can use the following trick to extend any \(X \in X(M)\) to \(X(U_\epsilon(M))|_M\).

First choose a smooth frame field \(E_1, ..., E_n\) defined along \(M\) so that \(E_i \in X(\overline{M})|_M\). We may arrange if need to have the first \(d\) of these tangent to \(M\). Now parallel translate each frame radially outward a distance \(\epsilon\) to obtain a smooth frame field \(\overline{E}_1, ..., \overline{E}_n\) on \(U_\epsilon(M)\).

Now we shall obtain a sort of splitting of the Levi-Civita connection of \(\overline{M}\) along the submanifold \(M\). The reader should recognize a familiar theme here
2.1. DEFINITIONS

especially if elementary surface theory is fresh in his or her mind. First we notice
that the Levi-Civita connection \( \nabla \) on \( M \) restrict nicely to a connection on the
bundle \( TM \rightarrow M \). The reader should be sure to realize that the space of
sections of this bundle is exactly \( \mathfrak{X}(M) \) and so the restricted connection is a
map \( \nabla|_M : \mathfrak{X}(M) \times \mathfrak{X}(M)|_M \rightarrow \mathfrak{X}(M)|_M \). The point is that if \( X \in \mathfrak{X}(M) \) and
\( W \in \mathfrak{X}(M)|_M \) then \( \nabla_X W \) doesn’t seem to be defined since \( X \) and \( W \) are not
elements of \( \mathfrak{X}(M) \). But we may extend \( X \) and \( W \) to elements of \( \mathfrak{X}(M) \) and then
restrict again to get an element of \( \mathfrak{X}(M)|_M \). Then recalling the local properties
of a connection we see that the result does not depend on the extension.

**Exercise 2.2** Use local coordinates to prove the claimed independence on the
extension.

We shall write simply \( \nabla \) in place of \( \nabla|_M \) since the context make it clear when
the later is meant. Thus \( \nabla_X W := \nabla_X W \) where \( X \) and \( W \) are any extensions
of \( X \) and \( W \) respectively.

Clearly we have \( \nabla_X (Y_1, Y_2) = \langle \nabla_X Y_1, Y_2 \rangle + \langle Y_1, \nabla_X Y_2 \rangle \) and so \( \nabla \) is a metric
connection on \( TM \). For a fixed \( X, Y \in \mathfrak{X}(M) \) we have the decomposition of
\( \nabla_X Y \) into tangent and normal parts. Similarly, for \( V \in \mathfrak{X}(M)^\perp \) we can consider
the decomposition of \( \nabla_X V \) into tangent and normal parts. Thus we have
\[
\nabla_X Y = (\nabla_X Y)^\tan + (\nabla_X Y)^\perp
\]
\[
\nabla_X V = (\nabla_X V)^\tan + (\nabla_X V)^\perp
\]
We make the following definitions:
\[
\nabla_X Y := (\nabla_X Y)^\tan \text{ for all } X, Y \in \mathfrak{X}(M)
\]
\[
b_{12}(X, Y) := (\nabla_X Y)^\perp \text{ for all } X, Y \in \mathfrak{X}(M)
\]
\[
b_{21}(X, V) := (\nabla_X V)^\tan \text{ for all } X \in \mathfrak{X}(M), V \in \mathfrak{X}(M)^\perp
\]
\[
\nabla_X V := (\nabla_X V)^\perp \text{ for all } X \in \mathfrak{X}(M), V \in \mathfrak{X}(M)^\perp
\]

Now if \( X, Y \in \mathfrak{X}(M) \), \( V \in \mathfrak{X}(M)^\perp \) then \( 0 = \langle Y, V \rangle \) and so
\[
0 = \nabla_X \langle Y, V \rangle
= \langle \nabla_X Y, V \rangle + \langle Y, \nabla_X V \rangle
= \langle (\nabla_X Y)^\perp, V \rangle + \langle Y, (\nabla_X V)^\tan \rangle
= \langle b_{12}(X, Y), V \rangle + \langle Y, b_{21}(X, V) \rangle.
\]
It follows that \( \langle b_{12}(X, Y), V \rangle = -\langle Y, b_{21}(X, V) \rangle \). Now we from this that \( b_{12}(X, Y) \)
is not only \( C^\infty(M) \) linear in \( X \) but also in \( Y \). Thus \( b_{12} \) is tensorial and so for
each fixed \( p \in M, b_{12}(X_p, Y_p) \) is a well defined element of \( T_p M^\perp \) for each fixed
\( X_p, Y_p \in T_p M \). Also, for any \( X_1, X_2 \in \mathfrak{X}(M) \) we have
\[
b_{12}(X_1, X_2) - b_{12}(X_2, X_1)
= (\nabla_{X_1} X_2 - \nabla_{X_2} X_1)^\perp
= ([X_1, X_2])^\perp = 0.
\]
So $b_{12}$ is symmetric. The classical notation for $b_{12}$ is $II$ and the form is called the second fundamental tensor or the second fundamental form. For $\xi \in T_p M$ we define the linear map $B_\xi(\cdot) := b_{12}(\xi, \cdot)$. With this in mind we can easily deduce the following facts which we list as a theorem:

1. $\nabla_X Y := (\nabla_X Y)^{\tan}$ defines a connection on $M$ which is identical with the Levi-Civita connection for the induced metric on $M$.

2. $(\nabla_X V)^\perp := \nabla_X^\perp V$ defines a metric connection on the vector bundle $TM^\perp$.

3. $(\nabla_X Y)^\perp := b_{12}(X, Y)$ defines a symmetric $C^\infty(M)$-bilinear form with values in $\mathfrak{X}(M)^\perp$.

4. $(\nabla_X Y)^{\tan} := b_{21}(X, V)$ defines a symmetric $C^\infty(M)$-bilinear form with values in $\mathfrak{X}(M)$.

**Corollary 2.3** $b_{21}$ is tensorial and so for each fixed $p \in M$, $b_{21}(X_p, Y_p)$ is a well defined element of $T_p M^\perp$ for each fixed $X_p, Y_p \in T_p M$ and we have a bilinear form $b_{21} : T_p M \times T_p M^\perp \rightarrow T_p M$.

**Corollary 2.4** The map $b_{21}(\xi, \cdot) : T_p M^\perp \rightarrow T_p M$ is equal to $-B_\xi^t : T_p M \rightarrow T_p M^\perp$.

Writing any $Y \in \mathfrak{X}(M)|_M$ as $Y = (Y^{\tan}, Y^\perp)$ we can write the map $\nabla_X : \mathfrak{X}(M)|_M \rightarrow \mathfrak{X}(M)|_M$ as a matrix of operators:

$$
\begin{bmatrix}
\nabla_X & B_X \\
-B_X^t & \nabla_X^\perp
\end{bmatrix}
$$

Next we define the shape operator which is also called the Weingarten map. There is not a perfect agreement on sign conventions; some author’s shape operator is the negative of the shape operator as defined by others. We shall define $S^+$ and $S^-$ which only differ in sign. This way we can handle both conventions simultaneously and the reader can see where the sign convention does or does not make a difference in any given formula.

**Definition 2.5** Let $p \in M$. For each unit vector $u$ normal to $M$ at $p$ we have a map called the (±) **shape operator** $S^\pm_u$ associated to $u$. defined by $S^\pm_u(v) := (\pm \nabla_v U)^{\tan}$ where $U$ is any unit normal field defined near $p$ such that $U(p) = u$.

The shape operators $\{S^\pm_u\} u$ a unit normal contain essentially the same information as the second fundamental tensor $II = b_{12}$. This is because for any $X, Y \in \mathfrak{X}(M)$ and $U \in \mathfrak{X}(M)^\perp$ we have

$$
\langle S^\pm_U X, Y \rangle = \langle (\pm \nabla_X U)^{\tan}, Y \rangle = \langle U, \pm \nabla_X Y \rangle \\
= \langle U, (\pm \nabla_X Y)^\perp \rangle = \langle U, \pm b_{12}(X, Y) \rangle \\
= \langle U, \pm II(X, Y) \rangle.
$$

Note: $\langle U, b_{12}(X, Y) \rangle$ is tensorial in $U, X$ and $Y$. Of course, $S^\pm_U X = -S^\pm_U X$. 
2.1. DEFINITIONS

Theorem 2.6 Let $M$ be a semi-Riemannian submanifold of $\bar{M}$. We have the Gauss equation

\[
(R_{VW}X,Y) = (\bar{R}_{VW}X,Y) - \langle II(V,X), II(W,Y) \rangle + \langle II(V,Y), II(W,X) \rangle
\]

Proof. Since this is clearly a tensor equation we may assume that $[V,W] = 0$. With this assumption we have we have $\langle \bar{R}_{VW}X,Y \rangle = \langle VW \rangle - \langle WV \rangle$ where $\langle VW \rangle = \langle \nabla V \nabla W X, Y \rangle$

\[
\langle \nabla V \nabla W X, Y \rangle = \langle \nabla V \nabla W X, Y \rangle + \langle \nabla V (II(W,X)), Y \rangle

= \langle \nabla V \nabla W X, Y \rangle + \langle \nabla V (II(W,X)), Y \rangle

= V \langle II(W,X), Y \rangle - \langle II(W,X), \nabla V Y \rangle

= \langle \nabla V \nabla W X, Y \rangle + V \langle II(W,X), Y \rangle - \langle II(W,X), \nabla V Y \rangle
\]

Since

\[
\langle II(W,X), \nabla V Y \rangle = \langle II(W,X), (\nabla V Y) \rangle = \langle II(W,X), II(V,Y) \rangle
\]

we have $\langle VW \rangle = \langle \nabla V \nabla W X, Y \rangle - \langle II(W,X), II(V,Y) \rangle$. Interchanging the roles of $V$ and $W$ and subtracting we get the desired conclusion.

Another formula that follows easily from the Gauss equation is the following formula (also called the Gauss formula):

\[
K(v \wedge w) = \bar{K}(v \wedge w) + \frac{\langle II(v,v), II(w,w) \rangle - \langle II(v,w), II(v,w) \rangle}{\langle v,v \rangle \langle w,w \rangle - \langle v,w \rangle^2}
\]

Exercise 2.7 Prove the last formula (second version of the Gauss equation).

From this last version of the Gauss equation we can show that a sphere $S^n(r)$ of radius $r$ in $\mathbb{R}^{n+1}$ has constant sectional curvature $1/r^2$ for $n > 1$. If $(u_i)$ is the standard coordinates on $\mathbb{R}^{n+1}$ then the position vector field in $\mathbb{R}^{n+1}$ is $r = \sum_{i=1}^{n+1} u_i \partial_i$. Recalling that the standard (flat connection) $D$ on $\mathbb{R}^{n+1}$ is just the Lie derivative we see that $D_X r = \sum_{i=1}^{n+1} X u_i \partial_i = X$. Now using the usual identifications, the unit vector field $U = r/r$ is the outward unit vector field on $S^n(r)$. We have

\[
\langle II(X,Y), U \rangle = \langle D_X Y, U \rangle

= \frac{1}{r} \langle D_X Y, r \rangle = -\frac{1}{r} \langle Y, D_X r \rangle

= -\frac{1}{r} \langle Y, X \rangle = -\frac{1}{r} \langle X, Y \rangle.
\]

Now letting $\bar{M}$ be $\mathbb{R}^{n+1}$ and $M$ be $S^n(r)$ and using the fact that the Gauss curvature of $\mathbb{R}^{n+1}$ is identically zero, the Gauss equation gives $K = 1/r$.

The second fundamental form contains information about how the semi-Riemannian submanifold $M$ bends about in $\bar{M}$. First we need a definition:
CHAPTER 2. SUBMANIFOLDS IN SEMI-RIEMANNIAN SPACES

Definition 2.8 Let $M$ be semi-Riemannian submanifold of $\bar{M}$ and $N$ a semi-Riemannian submanifold of $\bar{N}$. A pair isometry $\Phi : (\bar{M}, M) \to (\bar{N}, N)$ consists of an isometry $\Phi : M \to N$ such that $\Phi(M) = N$ and such that $\Phi|_M : M \to N$ is an isometry.

Proposition 2.9 A pair isometry $\Phi : (\bar{M}, M) \to (\bar{N}, N)$ preserves the second fundamental tensor:

$$T_p \Phi \cdot II(v, w) = II(T_p \Phi \cdot v, T_p \Phi \cdot w)$$

for all $v, w \in T_p M$ and all $p \in M$.

Proof. Let $p \in M$ and extend $v, w \in T_p M$ to smooth vector fields $V$ and $W$. Since an isometry respects the Levi-Civita connections we have $\Phi^* \bar{\nabla} V W = \bar{\nabla}_{\Phi_* V} \Phi_* W$. Now since $\Phi$ is a pair isometry we have $T_p \Phi(T_p M) \subset (T_{\Phi(p)} N)^\perp$. This means that $\Phi_* : X(\bar{M})|_M \to X(\bar{N})|_N$ preserves normal and tangential components $\Phi_* (X(\bar{M})) \subset X(N)$ and $\Phi_* (X(\bar{M})^\perp) \subset X(N)^\perp$. We have

$$T_p \Phi \cdot II(v, w) = \Phi_* II(V, W)(\Phi(p))$$

$$= \Phi_* (\nabla_\Phi V \cdot W) (\Phi(p))$$

$$= (\Phi_* \nabla_\Phi V \cdot W)(\Phi(p))$$

$$= II(\Phi_* V, \Phi_* W)(\Phi(p))$$

$$= II(T_p \Phi \cdot v, T_p \Phi \cdot w)$$

The following example is simple but conceptually very important.

Example 2.10 Let $M$ be the strip 2 dimensional strip $\{(x, y, 0) : -\pi < x < \pi\}$ considered as submanifold of $\mathbb{R}^3$ (with the canonical Riemannian metric). Let $N$ be the subset of $\mathbb{R}^3$ given by $\{(x, y, \sqrt{1-x^2}) : -1 < x < 1\}$. Exercise: Show that $M$ is isometric to $\bar{M}$. Show that there is no pair isometry $(\mathbb{R}^3, M) \to (\mathbb{R}^3, N)$.

2.2 Curves in Submanifolds

If $\gamma : I \to M$ is a curve in $M$ and $M$ is a semi-Riemannian submanifold of $\bar{M}$ then we have $\nabla_{\partial_t} Y = \nabla_{\partial_t} Y + II(\dot{\gamma}, Y)$ for any vector field $Y$ along $\gamma$. If $Y$ is a vector field in $\mathcal{X}(\bar{M})|_M$ or in $\mathcal{X}(\bar{M})$ then $Y \circ \gamma$ is a vector field along $\gamma$. In this case we shall still write $\nabla_{\partial_t} Y = \nabla_{\partial_t} Y + II(\dot{\gamma}, Y)$ rather than $\nabla_{\partial_t} (Y \circ \gamma) = \nabla_{\partial_t} (Y \circ \gamma) + II(\dot{\gamma}, Y \circ \gamma)$.
2.2. CURVES IN SUBMANIFOLDS

Recall that \( \dot{\gamma} \) is a vector field along \( \gamma \). We also have \( \ddot{\gamma} := \nabla_{\partial_t} \dot{\gamma} \) which in this context will be called the \textbf{extrinsic acceleration} (or acceleration in \( \bar{M} \)). By definition we have \( \nabla_{\partial_t} Y = (\nabla_{\partial_t} Y)^\perp \). The \textbf{intrinsic acceleration} (acceleration in \( M \)) is \( \nabla_{\partial_t} \dot{\gamma} \). Thus we have

\[
\ddot{\gamma} = \nabla_{\partial_t} \dot{\gamma} + II(\dot{\gamma}, \dot{\gamma}).
\]

From this definitions we can immediately see the truth of the following

**Proposition 2.11** If \( \gamma : I \rightarrow M \) is a curve in \( M \) and \( M \) is a semi-Riemannian submanifold of \( \bar{M} \) then \( \gamma \) is a geodesic in \( M \) if and only if \( \ddot{\gamma}(t) \) is normal to \( M \) for every \( t \in I \).

**Exercise 2.12** A constant speed parameterization of a great circle in \( S^n(r) \) is a geodesic. Every geodesic in \( S^n(r) \) is of this form.

**Definition 2.13** A semi-Riemannian manifold \( M \subset \bar{M} \) is called totally geodesic if every geodesic in \( M \) is a geodesic in \( \bar{M} \).

**Theorem 2.14** For a semi-Riemannian manifold \( M \subset \bar{M} \) the following conditions are equivalent

i) \( M \) is totally geodesic

ii) \( II \equiv 0 \)

iii) For all \( v \in TM \) the \( \bar{M} \) geodesic \( \gamma_v \) with initial velocity \( v \) is such that \( \gamma_v(0, \epsilon) \subset M \) for \( \epsilon > 0 \) sufficiently small.

iv) For any curve \( \alpha : I \rightarrow M \), parallel translation along \( \alpha \) induced by \( \nabla \) in \( \bar{M} \) is equal to parallel translation along \( \alpha \) induced by \( \nabla \) in \( M \).

**Proof.** (i)\( \Rightarrow \) (iii) follows from the uniqueness of geodesics with a given initial velocity.

(iii)\( \Rightarrow \) (ii): Let \( v \in TM \). Applying 2.11 to \( \gamma_v \) we see that \( II(v, v) = 0 \). Since \( v \) was arbitrary we conclude that \( II \equiv 0 \).

(ii)\( \Rightarrow \) (iv): Suppose \( v \in T_pM \). If \( V \) is a parallel vector field with respect to \( \nabla \) that is defined near \( p \) such that \( V(p) = v \). Then \( \nabla_{\partial_t} V = \nabla_{\partial_t} V + II(\dot{\gamma}, V) = 0 + 0 \) for any \( \gamma \) with \( \gamma(0) = p \) so that \( V \) is a parallel vector field with respect to \( \nabla \).

(iv)\( \Rightarrow \) (i): Assume (iv). If \( \gamma \) is a geodesic in \( M \) then \( \gamma' \) is parallel along \( \gamma \) with respect to \( \nabla \). Then by assumption \( \gamma' \) is parallel along \( \gamma \) with respect to \( \nabla \). Thus \( \gamma \) is also a \( \bar{M} \) geodesic. \( \blacksquare \)
2.3 Hypersurfaces

If the codimension of \( M \) in \( \bar{M} \) is equal to 1 then we say that \( M \) is a hypersurface. If \( M \) is a semi-Riemannian hypersurface in \( \bar{M} \) and \( \langle u, u \rangle > 0 \) for every \( u \in (T_p M)^\perp \) we call \( M \) a positive hypersurface. If \( \langle u, u \rangle < 0 \) for every \( u \in (T_p M)^\perp \) we call \( M \) a negative hypersurface. Of course, if \( \bar{M} \) is Riemannian then every hypersurface in \( \bar{M} \) is positive. The sign of \( M \), in \( \bar{M} \) denoted sgn \( M \) is sgn\( \langle u, u \rangle \).

Exercise 2.15 Suppose that \( c \) is a regular value of \( f \in C^\infty(\bar{M}) \) then \( M = f^{-1}(c) \) is a semi-Riemannian hypersurface if \( \langle df, df \rangle > 0 \) on all of \( M \) or if \( \langle df, df \rangle < 0 \). sgn\( \langle df, df \rangle \) = sgn\( \langle u, u \rangle \).

From the preceding exercise it follows if \( M = f^{-1}(c) \) is a semi-Riemannian hypersurface then \( U = \nabla f / \| \nabla f \| \) is a unit normal for \( M \) and \( \langle U, U \rangle = \text{sgn} \langle df, df \rangle \).

Notice that this implies that \( M = f^{-1}(c) \) is orientable if \( \bar{M} \) is orientable. Thus not every semi-Riemannian hypersurface is of the form \( f^{-1}(c) \). On the other hand every hypersurface is locally of this form.

In the case of a hypersurface we have (locally) only two choices of unit normal. Once we have chosen a unit normal \( u \) the shape operator is denoted simply by \( S \) rather than \( S_u \).

We are already familiar with the sphere \( S^n(r) \) which is \( f^{-1}(r^2) \) where \( f(x) = \langle x, x \rangle = \sum_{i=1}^n x^i x^i \). A similar example arises when we consider the semi-Euclidean space \( R^{n+1-\nu,\nu} \) where \( \nu \neq 0 \). In this case, the metric is \( \langle x, y \rangle = -\sum_{i=1}^\nu x^i y^i + \sum_{i=\nu+1}^n x^i y^i \). We let \( f(x) := -\sum_{i=1}^\nu x^i x^i + \sum_{i=\nu+1}^n x^i x^i \) and then for \( r > 0 \) we have that \( f^{-1}(r^2) \) is a semi-Riemannian hypersurface in \( R^{n+1-\nu,\nu} \) with sign \( \varepsilon \) and unit normal \( U = x/r \). We shall divide these hypersurfaces into two classes according to sign.

**Definition 2.16** For \( n > 1 \) and \( 0 \leq \nu \leq n \), we define
\[
S^n_\nu(r) = \{ x \in R^{n+1-\nu,\nu} : \langle x, x \rangle = r^2 \}.
\]

\( S^n_\nu(r) \) is called the pseudo-sphere of index \( \nu \).

**Definition 2.17** For \( n > 1 \) and \( 0 \leq \nu \leq n \), we define
\[
H^n_\nu(r) = \{ x \in R^{n+1-(\nu+1),\nu+1} : \langle x, x \rangle = -r^2 \}.
\]

\( H^n_\nu(r) \) is called the pseudo-hyperbolic space of radius \( r \) and index \( \nu \).
Chapter 3

Lie groups II

3.1 Lie Group Actions

The basic definitions for group actions were given earlier in definition ?? and ???. As before we give most of our definitions and results for left actions and ask the reader to notice that analogous statements can be made for right actions.

Definition 3.1 Let \( l : G \times M \rightarrow M \) be a left action where \( G \) is a Lie group and \( M \) a smooth manifold. If \( l \) is a smooth map then we say that \( l \) is a (smooth) Lie group action.

As before, we also use either of the notations \( gp \) or \( l_g(p) \) for \( l(g, p) \). For right actions \( r : M \times G \rightarrow M \) we write \( pg = r_g(p) = r(p, g) \). A right action corresponds to a left action by the rule \( gp := pg^{-1} \). Recall that for \( p \in M \) the orbit of \( p \) is denoted \( Gp \) and we call the action transitive if \( Gp = M \).

Definition 3.2 Let \( l \) be a Lie group action as above. For a fixed \( p \in M \) the isotropy group of \( p \) is defined to be

\[
G_p := \{ g \in G : gp = p \}
\]

The isotropy group of \( p \) is also called the stabilizer of \( p \).

Exercise 3.3 Show that \( G_p \) is a closed subset and abstract subgroup of \( G \). This means that \( G_p \) is a closed Lie subgroup.

Recalling the definition of a free action, it is easy to see that an action is free if and only if the isotropy subgroup of every point is the trivial subgroup consisting of the identity element alone.

Definition 3.4 Suppose that we have Lie group action of \( G \) on \( M \). If \( N \) is a subset of \( M \) and \( gx \in x \) for all \( x \in N \) then we say that \( N \) is an invariant subset. If \( N \) is also a submanifold then it is called an invariant submanifold.
In this definition we include the possibility that \( N \) is an open submanifold. If \( N \) is an invariant subset of \( N \) then it is easy to set that \( gN = N \) where \( gN = l_g(N) \) for any \( g \). Furthermore, if \( N \) is a submanifold then the action restricts to a Lie group action \( G \times N \to N \).

If \( G \) is zero dimensional then by definition it is just a group with discrete topology and we recover the definition of discrete group action. We have already seen several examples of discrete group actions and now we list a few examples of more general Lie group actions.

**Example 3.5** In case \( M = \mathbb{R}^n \) then the Lie group \( GL(n, \mathbb{R}) \) acts on \( \mathbb{R}^n \) by matrix multiplication. Similarly, \( GL(n, \mathbb{C}) \) acts on \( \mathbb{C}^n \). More abstractly, \( GL(V) \) acts on the vector space \( V \). This action is smooth since \( Ax \) depends smoothly (polynomially) on the components of \( A \) and on the components of \( x \in \mathbb{R}^n \).

**Example 3.6** Any Lie subgroup of \( GL(n, \mathbb{R}) \) acts on \( \mathbb{R}^n \) also by matrix multiplication. For example, \( O(n, \mathbb{R}) \) acts on \( \mathbb{R}^n \). For every \( x \in \mathbb{R}^n \) the orbit of \( x \) is the sphere of radius \( \|x\| \). This is trivially true if \( \|x\| = 0 \). In general, if \( \|x\| \neq 0 \) then, \( \|gx\| = \|x\| \) for any \( g \in O(n, \mathbb{R}) \). On the other hand, if \( x, y \in \mathbb{R}^n \) and \( \|x\| = \|y\| = r \) then let \( \tilde{x} := x/r \) and \( \tilde{y} := y/r \). Extend to orthonormal bases \( (\tilde{x} = e_1, \ldots, e_n) \) and \( (\tilde{y} = f_1, \ldots, f_n) \). Then there exists an orthogonal matrix \( S \) such that \( Se_i = f_i \) for \( i = 1, \ldots, n \). In particular, \( S\tilde{x} = \tilde{y} \) and so \( Sx = y \).

**Exercise 3.7** From the last example we can restrict the action of \( O(n, \mathbb{R}) \) to a transitive action on \( S^{n-1} \). Now \( SO(n, \mathbb{R}) \) also acts on \( \mathbb{R}^n \) and by restriction on \( S^{n-1} \). From the last example we know that \( O(n, \mathbb{R}) \) acts transitively on \( S^{n-1} \). Show that the same is true for \( SO(n, \mathbb{R}) \) as long as \( n > 1 \).

A Lie group acts on itself in an obvious way:

**Definition 3.8** For a Lie group \( G \) and a fixed element \( g \in G \), the maps \( L_g : G \to G \) and \( R_g : G \to G \) are defined by

\[
L_g x = gx \quad \text{for} \quad x \in G \\
R_g x = xg \quad \text{for} \quad x \in G
\]

and are called left translation and right translation (by \( g \)) respectively.

The maps \( G \times G \to G \) given by \((g, x) \mapsto L_gx \) and \((g, x) \mapsto R_g x \) are Lie group actions.

**Example 3.9** If \( H \) is a Lie subgroup of a Lie group \( G \) then we can consider \( L_h \) for any \( h \in H \) and thereby obtain an action of \( H \) on \( G \).

Recall that a subgroup \( H \) of a group \( G \) is called a normal subgroup if \( gkg^{-1} \in K \) for any \( k \in H \) and all \( g \in G \). In other word, \( H \) is normal if \( gHg^{-1} \subset H \) for all \( g \in G \) and it is easy to see that in this case we always have \( gHg^{-1} = H \).
### Example 3.10
If $H$ is a normal Lie subgroup of $G$, then $G$ acts on $H$ by conjugation:

\[ C_g h = ghg^{-1} \]

Suppose now that a Lie group $G$ acts on smooth manifolds $M$ and $N$. For simplicity we take both actions to be left action which we denote by $l$ and $\lambda$ respectively. A map $\Phi: M \to N$ such that $\Phi \circ l_g = \lambda_g \circ \Phi$ for all $g \in G$, is said to be an equivariant map (equivariant with respect to the given actions). This means that for all $g$ the following diagram commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{\Phi} & N \\
l_g \downarrow & & \lambda_g \\
M & \xrightarrow{\Phi} & N
\end{array}
\]

If $\Phi$ is also a diffeomorphism then we have an equivalence of Lie group actions.

### Example 3.11
If $\phi: G \to H$ is a Lie group homomorphism then we can define an action of $G$ on $H$ by $\lambda(g, h) = L_{\phi(g)}h$. We leave it to the reader to verify that this is indeed a Lie group action. In this situation $\phi$ is equivariant with respect to the actions $\lambda$ and $L$ (left translation).

### Example 3.12
Let $T^n = S^1 \times \cdots \times S^1$ be the $n$-torus where we identify $S^1$ with the complex numbers of unit modulus. Fix $k = (k_1, \ldots, k_n) \in \mathbb{R}^n$. Then $\mathbb{R}$ acts on $\mathbb{R}^n$ by $\tau^k(t, x) = t \cdot x := x + tk$. On the other hand, $\mathbb{R}$ acts on $T^n$ by $t \cdot (z^1, \ldots, z^n) = (e^{itk_1}z^1, \ldots, e^{itk_n}z^n)$. The map $\mathbb{R}^n \to T^n$ given by $(x^1, \ldots, x^n) \mapsto (e^{ix^1}, \ldots, e^{ix^n})$ is equivariant with respect to these actions.

### Theorem 3.13 (Equivariant Rank Theorem)
Suppose that $f: M \to N$ is smooth and that a Lie group $G$ acts on both $M$ and $N$ with the action on $M$ being transitive. If $f$ is equivariant then it has constant rank. In particular, each level set of $f$ is a closed regular submanifold.

**Proof.** Let the actions on $M$ and $N$ be denoted by $l$ and $\lambda$ respectively as before. Pick any two points $p_1, p_2 \in M$. Since $G$ acts transitively on $M$ there is a $g$ with $l_g p_1 = p_2$. By hypothesis, we have the following commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
l_g \downarrow & & \lambda_g \\
M & \xrightarrow{f} & N
\end{array}
\]

which, upon application of the tangent functor gives the commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{T_{\phi}f} & N \\
T_p l_g \downarrow & & \downarrow T_{\lambda_g}f \\
M & \xrightarrow{T_{\phi}f} & N
\end{array}
\]
Since the maps $T_p l_g$ and $T_{f(p_1)} \lambda_g$ are linear isomorphisms we see that $T_p f$ must have the same rank as $T_{p_1} f$. Since $p_1$ and $p_2$ were arbitrary we see that the rank of $f$ is constant on $M$. 

There are several corollaries of this neat theorem. For example, we know that $O(n, \mathbb{R})$ is the level set $f^{-1}(I)$ where $f : GL(n, \mathbb{R}) \to gl(n, \mathbb{R}) = M(n, \mathbb{R})$ is given by $f(A) = A^T A$. The group $O(n, \mathbb{R})$ acts on itself via left translation and we also let $O(n, \mathbb{R})$ act on $gl(n, \mathbb{R})$ by $Q \cdot A := Q^T A Q$ (adjoint action). One checks easily that $f$ is equivariant with respect to these actions and since the first action (left translation) is certainly transitive we see that $O(n, \mathbb{R})$ is a closed regular submanifold of $GL(n, \mathbb{R})$. It follows from proposition ?? that $O(n, \mathbb{R})$ is a closed Lie subgroup of $GL(n, \mathbb{R})$. Similar arguments apply for $U(n, \mathbb{C}) \subset GL(n, \mathbb{C})$ and other linear Lie groups. In fact we have the following general corollary to Theorem 3.13 above.

**Corollary 3.14** If $\phi : G \to H$ is a Lie group homomorphism then the kernel $\text{Ker}(h)$ is a closed Lie subgroup of $G$.

**Proof.** Let $G$ act on itself and on $H$ as in example 3.11. Then $\phi$ is equivariant and $\phi^{-1}(e) = \text{Ker}(h)$ is a closed Lie subgroup by Theorem 3.13 and Proposition ??.

We also have use for the

**Corollary 3.15** Let $l : G \times M \to M$ be a Lie group action and $G_p$ the isotropy subgroup of some $p \in M$. Then $G_p$ is a closed Lie subgroup of $G$.

**Proof.** The orbit map $\theta_p : G \to M$ given by $\theta_p(g) = gp$ is equivariant with respect to left translation on $G$ and the given action on $M$. Thus by the equivariant rank theorem, $G_p$ is a regular submanifold of $G$ an then by Proposition ?? it is a closed Lie subgroup.

### 3.1.1 Proper Lie Group Actions

**Definition 3.16** Let $l : G \times M \to M$ be a smooth (or merely continuous) group action. If the map $P : G \times M \to M \times M$ given by $(g, p) \mapsto (l_g p, p)$ is proper we say that the action is a **proper action**.

It is important to notice that a proper action is *not* defined to be an action such that the defining map $l : G \times M \to M$ is proper.

We now give a useful characterization of a proper action. For any subset $K \subset M$, let $g \cdot K := \{gx : x \in K\}$.

**Proposition 3.17** Let $l : G \times M \to M$ be a smooth (or merely continuous) group action. Then $l$ is a proper action if and only if the set $G_K := \{g \in G : (g \cdot K) \cap K \neq \emptyset\}$ is compact whenever $K$ is compact.
3.1. LIE GROUP ACTIONS

**Proof.** We follow the proof from [?];

Suppose that \( l \) is proper so that the map \( P \) is a proper map. Let \( \pi_G \) be the first factor projection \( G \times M \to G \). Then

\[
G_K = \{ g : \text{there exists a } x \in K \text{ such that } gx \in K \} \\
= \{ g : \text{there exists a } x \in M \text{ such that } P(g, x) \in K \times K \} \\
= \pi_G(P^{-1}(K \times K))
\]

and so \( G_K \) is compact.

Next we assume that \( G_K \) is compact for all compact \( K \). If \( C \subset M \times M \) is compact then letting \( K = \pi_1(C) \cap \pi_2(C) \) where \( \pi_1 \) and \( \pi_2 \) are first and second factor projections \( M \times M \to M \) respectively we have

\[
P^{-1}(C) \subset P^{-1}(K \times K) \subset \{(g, x) : gp \in K\} \\
\subset G_K \times K.
\]

Since \( P^{-1}(C) \) is a closed subset of the compact set \( G_K \times K \) it is compact. This means that \( P \) is proper since \( C \) was an arbitrary compact subset of \( M \times M \).

Using this proposition, one can show that definition ?? for discrete actions is consistent with definition 3.16 above.

**Proposition 3.18** If \( G \) is compact then any smooth action \( l : G \times M \to M \) is proper.

**Proof.** Let \( K \subset M \times M \) be compact. We find compact \( C \subset M \) such that \( K \subset C \times C \) as in the proof of proposition 3.17.

Claim: \( P^{-1}(K) \) is compact. Indeed,

\[
P^{-1}(K) \subset P^{-1}(C \times C) = \cup_{c \in C} P^{-1}(C \times \{c\}) \\
= \cup_{c \in C} \{(g, p) : (gp, p) \in C \times \{c\}\} \\
= \cup_{c \in C} \{(g, c) : gp \in C\} \\
\subset \cup_{c \in C} (G \times \{c\}) = G \times C
\]

Thus \( P^{-1}(K) \) is a closed subset of the compact set \( G \times C \) and hence is compact.

**Exercise 3.19** Prove the following

i) If \( l : G \times M \to M \) is a proper action and \( H \subset G \) is a closed subgroup then the restricted action \( H \times M \to M \) is proper.

ii) If \( N \) is an invariant submanifold for a proper action \( l : G \times M \to M \) then the restricted action \( G \times N \to N \) is also proper.

Let us now consider a Lie group action \( l : G \times M \to M \) that is both proper and free. The map orbit map at \( p \) is the map \( \theta_p : G \to M \) given by \( \theta_p(g) = g \cdot p \). It is easily seen to be smooth and its image is obviously \( G \cdot p \). In
fact, if the action is free then each orbit map is injective. Also, $\theta_p$ is equivariant with respect to the left action of $G$ on itself and the action $l$:

$$\theta_p(gx) = (gx) \cdot p = g \cdot (x \cdot p)$$

$$= g \cdot \theta_p(x)$$

for all $x, g \in G$. It now follows from Theorem 3.13 (the equivariant rank theorem) that $\theta_p$ has constant rank and since it is injective it must be an immersion. Not only that, but it is a proper map. Indeed, for any compact $K \subset M$ the set $\theta_p^{-1}(K)$ is a closed subset of the set $G_{K \cup \{p\}}$ and since the later set is compact by Theorem 3.17, $\theta_p^{-1}(K)$ is compact. Now by exercise 3.1 obtain the result that $\theta_p$ is an embedding and each orbit is a regular submanifold of $M$. It will be very convenient to have charts on $M$ which fit the action of $G$ in a nice way. See figure 3.1.

**Definition 3.20** Let $M$ be an $n$-manifold and $G$ a Lie group of dimension $k$. If $l : G \times M \to M$ is a Lie group action then an action-adapted chart on $M$ is a chart $(U, x)$ such that

i) $x(U)$ is a product open set $V_1 \times V_1 \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$

ii) if an orbit has nonempty intersection with $U$ then that intersection has the form

$$\{x^{k+1} = c^1, ..., x^n = c^{n-k}\}$$

for some constants $c^1, ..., c^{n-k}$.

**Theorem 3.21** If $l : G \times M \to M$ is a free and proper Lie group action then for every $p \in M$ there is an action-adapted chart centered at $p$.

**Proof.** Let $p \in M$ be given. Since $G \cdot p$ is a regular submanifold we may choose a regular submanifold chart $(W, y)$ centered at $p$ so that $(G \cdot p) \cap W$ is exactly given by $y^{k+1} = ... = y^n = 0$ in $W$. Let $S$ be the complementary slice in $W$ given by $y^1 = ... = y^k = 0$. Note that $S$ is a regular submanifold. The tangent space $T_p M$ decomposes as

$$T_p M = T_p(G \cdot p) \oplus T_p S$$

Let $\varphi : G \times S \to M$ be the restriction of the action $l$ to the set $G \times S$. Also, let $i_p : G \to G \times S$ be the insertion map $g \mapsto (g, p)$ and let $j_c : S \to G \times S$ be the insertion map $s \mapsto (c, s)$. These insertion maps are embeddings and we have $\theta_p = \varphi \circ i_p$ and also $\varphi \circ j_c = \iota$ where $\iota$ is the inclusion $S \hookrightarrow M$. Now $T_{e \cdot} \theta_p(T_e G) = T_p(G \cdot p)$ since $\theta_p$ is an embedding. On the other hand, $T\varphi = T\varphi \circ Ti_p$ and so the image of $T_{(e, p)} \varphi$ must contain $T_p(G \cdot p)$. Similarly, from the composition $\varphi \circ j_c = \iota$ we see that the image of $T_{(e, p)} \varphi$ must contain $T_p S$. It follows that $T_{(e, p)} \varphi : T_{(e, p)}(G \times S) \to T_p M$ is surjective and since $T_{(e, p)}(G \times S)$ and $T_p M$ have the same dimension it is also injective.
By the inverse function theorem, there is neighborhood $O$ of $(e, p)$ such that $\varphi|O$ is a diffeomorphism. By shrinking $O$ further if necessary we may assume that $\varphi(O) \subset W$. We may also arrange that $O$ has the form of a product $O = A \times B$ for $A$ open in $G$ and $B$ open in $S$. In fact, we can assume that there are diffeomorphisms $\alpha : I^k \to A$ and $\beta : I^{n-k} \to B$ where $I^k$ and $I^{n-k}$ are the open cubes in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ given respectively by $I^k = (-1,1)^k$ and $I^{n-k} = (-1,1)^{n-k}$ and where $\alpha(e) = 0 \in \mathbb{R}^k$ and $\beta(p) = 0 \in \mathbb{R}^{n-k}$. Let $U := \varphi(A \times B)$. The map $\varphi \circ (\alpha \times \beta) : I^k \times I^{n-k} \to U$ is a diffeomorphism and so its inverse is a chart. We must make one more adjustment. We must show that $B$ can be chosen small enough that the intersection of each orbit with $B$ is either empty or a single point. If this were not true then there would be a sequence of open sets $B_i$ with compact closure (and with corresponding diffeomorphisms $\beta_i : I^k \to B_i$ as above) such that for every $i$ there is a pair of distinct points $p_i, p'_i \in B_i$ with $g_i p_i = p'_i$ for some sequence $\{g_i\} \subset G$. Since manifolds are first countable and normal, we may assume that the sequence $\{B_i\}$ is a nested neighborhood basis which means that $B_{i+1} \subset B_i$ for all $i$ and for each neighborhood $V$ of $p$, we have $B_i \subset V$ for large enough $i$. This forces both $p_i$ and $p'_i = g_i p_i$ to converge to $p$. From this we see that the set $K = \{(g_i p_i, p_i), (p, p)\} \subset M \times M$ is compact. Recall that by definition the map $P : (g, x) \mapsto (gx, x)$ is proper. Since $(g_i, p_i) = P^{-1}(g p_i, p_i)$ we see that $\{(g_i, p_i)\}$ is a subset of the compact set $P^{-1}(K)$. Thus after passing to a subsequence we have that $(g_i, p_i)$ converges to $(g, p)$ for some $g$ and hence
$g_i \to g$ and $g_ip_i \to gp$. But this means we have

$$gp = \lim_{i \to \infty} g_ip_i = \lim_{i \to \infty} p'_i = p$$

and since the action is free we conclude that $g = e$. But this is impossible since it would mean that for large enough $i$ we would have $g_i \in A$ and in turn this would imply that

$$\varphi(g_i, p_i) = l_{g_i}(p_i) = p'_i = l_e(p'_i) = \varphi(e, p'_i)$$

contradicting the injectivity of $\varphi$ on $A \times B$. Thus after shrinking $B$ we may assume that the intersection of each orbit with $B$ is either empty or a single point. We leave it to the reader to check that with $x := (\varphi \circ (\alpha \times \beta))^{-1} : U \to I^k \times I^{n-k} \subset \mathbb{R}^n$ we obtain a chart $(U, x)$ with the desired properties. ■

For the next lemma we continue with the convention that $I$ is the interval $(-1, 1)$.

**Lemma 3.22** Let $x := (\varphi \circ (\alpha \times \beta))^{-1} : U \to I^k \times I^{n-k} = I^n \subset \mathbb{R}^n$ be an action-adapted chart map obtained as in the proof of Theorem 3.21 above. Then given any $p_1 \in U$, there exists a diffeomorphism $\psi : I^n \to I^n$ such that $\psi \circ x$ is an action-adapted chart centered at $p_1$.

**Proof.** Clearly all we need to do is show that for any $a \in I^n$ there is a diffeomorphism $\psi : I^n \to I^n$ such that $\psi(a) = 0$. Let $a^i$ be the $i$–th component of $a$. Let $\psi_i : I \to I$ be defined by

$$\psi_i := \phi \circ l_{-\phi(a^i)} \circ \phi$$
where \( t_\cdot c(x) := x - c \) and \( \phi : (-1, 1) \to \mathbb{R} \) is the useful diffeomorphism \( \phi : x \mapsto \tan(\pi x) \). The diffeomorphism we want is now \( \psi(x) = (\psi_1(x_1), \ldots, \psi_n(x_n)) \).

### 3.1.2 Quotients

If \( l : G \times M \to M \) is a Lie group action, then there is a natural equivalence relation on \( M \) whereby the equivalence classes are exactly the orbits of the action. The quotient space (or orbit space) is denoted \( G \backslash M \) and we have the quotient map \( \pi : M \to G \backslash M \) so that \( A \subset G \backslash M \) is open if and only if \( \pi^{-1}(A) \) is open in \( M \). The quotient map is also open. Indeed, let \( U \subset M \) be open. We want to show that \( \pi(U) \) is open and for this it suffices to show that \( \pi^{-1}(\pi(U)) \) is open. But \( \pi^{-1}(\pi(U)) \) is the union \( \bigcup G_\cdot l_g(U) \) and this is open since each \( l_g(U) \) is open.

**Lemma 3.23** \( G \backslash M \) is a Hausdorff space if the set \( \Gamma := \{(gp, p) : g \in G, p \in M \} \) is a closed subset of \( M \times M \).

**Proof.** Let \( p, q \in G \backslash M \) with \( \pi(p) = p \) and \( \pi(q) = q \). If \( p \neq q \) then \( p \) and \( q \) are not in the same orbit. This means that \( (p, q) \notin \Gamma \) and so there must be a product open set \( U \times V \) such that \( (p, q) \in U \times V \) and \( U \times V \) disjoint from \( \Gamma \). This means that \( \pi(U) \) and \( \pi(V) \) are disjoint neighborhoods of \( p \) and \( q \) respectively.

**Proposition 3.24** If \( l : G \times M \to M \) is a free and proper action then \( G \backslash M \) is Hausdorff and paracompact.

**Proof.** To show that \( G \backslash M \) is Hausdorff we use the previous lemma. We must show that \( \Gamma \) is closed. But \( \Gamma = P(G \times M) \) is closed since \( P \) is proper.

To show that \( G \backslash M \) is paracompact it suffices to show that each connected component of \( G \backslash M \) is second countable. This reduces the situation to the case where \( G \backslash M \) is connected. In this case we can see that if \( \{U_i\} \) is a countable basis for the topology on \( M \) then \( \{\pi(U_i)\} \) is a countable basis for the topology on \( G \backslash M \).

We will shortly show that if the action is free and proper then \( G \backslash M \) has a smooth structure which makes the quotient map \( \pi : M \to G \backslash M \) a submersion. Before coming to this let us note that if such a smooth structure exists then it is unique. Indeed, if \( (G \backslash M)_A \) is \( G \backslash M \) with a smooth structure given by maximal atlas \( A \) and similarly for \( (G \backslash M)_B \) for another atlas \( B \) then we have the following commutative diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{\pi} & (G \backslash M)_A \\
\downarrow & & \downarrow \pi \\
(G \backslash M)_A & \xrightarrow{id} & (G \backslash M)_B
\end{array}
\]

Since \( \pi \) is a surjective submersion, Proposition ?? applies to show that \( (G \backslash M)_A \xrightarrow{id} (G \backslash M)_B \) is smooth as is its inverse. This means that \( A = B \).
Theorem 3.25 If \( l : G \times M \rightarrow M \) is a free and proper Lie group action then there is a unique smooth structure on the quotient \( G \backslash M \) such that

(i) the induced topology is the quotient topology and hence \( G \backslash M \) is a smooth manifold,

(ii) the projection \( \pi : M \rightarrow G \backslash M \) is a submersion,

(iii) \( \dim(G \backslash M) = \dim(M) - \dim(G) \).

Proof. Let \( \dim(M) = n \) and \( \dim(G) = k \). We have already shown that \( G \backslash M \) is a paracompact Hausdorff space. All that is left is to exhibit an atlas such that the charts are homeomorphisms with respect to this quotient topology. Let \( q \in G \backslash M \) and choose \( p \) with \( \pi(p) = q \). Let \( (U, x) \) be an action-adapted chart centered at \( p \) and constructed exactly as in Theorem 3.21. Let \( \pi(U) = V \subset G \backslash M \) and let \( B \) be the slice \( x^1 = \cdots = x^k = 0 \). By construction \( \pi|_B : B \rightarrow V \) is a bijection and in fact it is easy to check that \( \pi|_B \) is a homeomorphism and \( \sigma := (\pi|_B)^{-1} \) is the corresponding local section. Consider the map \( y = \pi_2 \circ x \circ \sigma \) where \( \pi_2 \) is the second factor projection \( \pi_2 : \mathbb{R}^k \times \mathbb{R}^{n-k} \rightarrow \mathbb{R}^{n-k} \). This is a homeomorphism since \( (\pi_2 \circ x)|_B \) is a homeomorphsim and \( \pi_2 \circ x \circ \sigma = (\pi_2 \circ x)|_B \circ \sigma \). We now have chart \( (V, y) \).

Given two such charts \( (V, y) \) and \( (\bar{V}, \bar{y}) \) we must show that \( \bar{y}^{-1} \circ y^{-1} \) is smooth. The \( (V, y) \) and \( (\bar{V}, \bar{y}) \) are constructed from associated action-adapted charts \( (U, x) \) and \( (\bar{U}, \bar{x}) \) on \( M \). Let \( q \in V \cap \bar{V} \). As in the proof of Lemma 3.22 we may find diffeomorphisms \( \psi \) and \( \bar{\psi} \) so that \( (U, \psi \circ x) \) and \( (\bar{U}, \bar{\psi} \circ \bar{x}) \) are action-adapted charts centered at points \( p_1 \in \pi^{-1}(q) \) and \( p_2 \in \pi^{-1}(q) \) respectively. Correspondingly, the charts \( (V, y) \) and \( (\bar{V}, \bar{y}) \) are modified to charts \( (V, y_\psi) \) and \( (\bar{V}, \bar{y}_{\bar{\psi}}) \) centered at \( q \) where

\[
\begin{align*}
y_\psi := \pi_2 \circ \psi \circ x \circ \sigma \\
\bar{y}_{\bar{\psi}} := \pi_2 \circ \bar{\psi} \circ \bar{x} \circ \sigma
\end{align*}
\]

One checks \( y_\psi \circ y^{-1} = \pi_2 \circ \psi \) and similarly for \( \bar{y}_{\bar{\psi}} \circ \bar{y}^{-1} \). From this it follows that the overlap map \( \bar{y}_{\bar{\psi}}^{-1} \circ y_\psi \) will be smooth if and only if \( \bar{y}^{-1} \circ y^{-1} \) is smooth. Thus we have reduced to the case where \( (U, x) \) and \( (\bar{U}, \bar{x}) \) are centered at \( p_1 \in \pi^{-1}(q) \) and \( p_2 \in \pi^{-1}(q) \) respectively. This entails that both \( (V, y) \) and \( (\bar{V}, \bar{y}) \) are centered at \( q \in V \cap \bar{V} \). Now if we choose a \( g \in G \) such that \( l_q(p_1) = p_2 \) then by composing with the diffeomorphism \( l_q \) we can reduce further to the case where \( p_1 = p_2 \). Here we use the fact that \( l_q \) takes the set of orbits to the set of orbits in a bijective manner and the special nature of our action-adapted charts with respect to these orbits. In this case the overlap map \( \bar{x} \circ x^{-1} \) must have the form \( (a, b) \mapsto (f(a, b), g(b)) \) for some smooth functions \( f \) and \( g \). It follows that \( \bar{y}^{-1} \circ y^{-1} \) has the form \( b \mapsto g(b) \).

Similar results hold for right actions. In fact, some of the most important examples of proper actions are usually presented as right actions (the right action associated to a principal bundle). In fact, we shall see situations where there is both a right and a left action in play.
Example 3.26 Consider $S^{2n-1}$ as the subset of $\mathbb{C}^n$ given by $S^{2n-1} = \{\xi \in \mathbb{C}^n : |\xi| = 1\}$. Here $\xi = (z^1, ..., z^n)$ and $|\xi| = \sum z^i \bar{z}^i$. Now we let $S^1$ act on $S^{2n-1}$ by $(a, \xi) \mapsto a\xi = (az^1, ..., az^n)$. This action is free and proper. The quotient is the complex projective space $\mathbb{C}P^{n-1}$.

$$S^{2n-1} \xrightarrow{\sim} \mathbb{C}P^{n-1}$$

These maps (one for each $n$) are called the Hopf maps. In this context $S^1$ is usually denoted by $U(1)$.

In the sequel we will be considering the similar right action $S^n \times U(1) \to S^n$. In this case we think of $\mathbb{C}P^n$ as consisting of column vector and the action is given by $(\xi, a) \mapsto a\xi$. Of course, since $U(1)$ is abelian this makes essentially no difference but in the next example we consider the quaternionic analogue where keeping track of order is important.

The quaternionic projective $\mathbb{H}P^{n-1}$ space is defined by analogy with $\mathbb{C}P^{n-1}$. The elements of $\mathbb{H}P^{n-1}$ are 1-dimensional subspaces of the right $\mathbb{H}$-vector space $\mathbb{H}^n$. Let us refer to these as $\mathbb{H}$-lines. Each of these are of real dimension 4. Each element of $\mathbb{H}^n \setminus \{0\}$ determines an $\mathbb{H}$-line and the $\mathbb{H}$-line determined by $(\xi^1, ..., \xi^n)\ell$ will be the same as that determined $(\xi^1, ..., \xi^n)\ell$ if and only if there is a nonzero element $a \in \mathbb{H}$ so that $(\xi^1, ..., \xi^n)\ell = (\xi^1, ..., \xi^n)\ell = (\xi^1a, ..., \xi^n a)\ell$. This defines an equivalence relation $\sim$ on $\mathbb{H}^n \setminus \{0\}$ and thus we may also think of $\mathbb{H}P^{n-1}$ as $(\mathbb{H}^n \setminus \{0\}) / \sim$. The element of $\mathbb{H}P^{n-1}$ determined by $(\xi^1, ..., \xi^n)\ell$ is denoted by $[\xi^1, ..., \xi^n]$. Notice that the subset $\{\xi \in \mathbb{H}^n : |\xi| = 1\}$ is $S^{4n-1}$. But as for the complex projective spaces we observe that all such $\mathbb{H}$-lines contain points of $S^{4n-1}$ and two points $\xi, \zeta \in S^{4n-1}$ determine the same $\mathbb{H}$-line if and only if $\xi = \zeta a$ for some $a$ with $|a| = 1$. Thus we can think of $\mathbb{H}P^{n-1}$ as a quotient of $S^{4n-1}$. When viewed in this way, we also denote the equivalence class of $\xi = (\xi^1, ..., \xi^n)\ell \in S^{4n-1}$ by $[\xi] = [\xi^1, ..., \xi^n]$. The equivalence classes are clearly the orbits of an action as described in the following example.

Example 3.27 Consider $S^{4n-1}$ considered as the subset of $\mathbb{H}^n$ given by $S^{4n-1} = \{\xi \in \mathbb{H}^n : |\xi| = 1\}$. Here $\xi = (\xi^1, ..., \xi^n)\ell$ and $|\xi| = \sum \xi \bar{\xi}$. Now we define a right action of $U(1, \mathbb{H})$ on $S^{4n-1}$ by $(\xi, a) \mapsto a\xi = (\xi^1a, ..., \xi^n a)\ell$. This action is free and proper. The quotient is the quaternionic projective space $\mathbb{H}P^{n-1}$ and we have the quotient map denoted by $\phi$

$$S^{4n-1} \xrightarrow{\phi} \mathbb{H}P^{n-1}$$

This map is also referred to as a Hopf map. Recall that $\mathbb{Z}_2 = \{1, -1\}$ acts on $S^{n-1} = \mathbb{R}^n$ on the right (or left) by multiplication and the action is a (discrete) proper and free action with quotient $\mathbb{R}P^{n-1}$ and so the above two examples generalize this.
For completeness we describe an atlas for \( \mathbb{H}P^{n-1} \). View \( \mathbb{H}P^{n-1} \) as the quotient \( S^{4n-1}/\sim \) described above. Let

\[ U_k := \{ [\xi] \in S^{4n-1} \subset \mathbb{H}^n : \xi_k \neq 0 \} \]

and define \( \varphi_k : U_k \rightarrow \mathbb{H}^{n-1} \cong \mathbb{R}^{4n-1} \) by

\[ \varphi_k([\xi]) = (\varphi_i([\xi_1, ..., \xi_n]) = (\xi_1 \xi_1^{-1}, ..., \hat{1}, ..., \xi_n \xi_n^{-1}) \]

where as for the real and complex cases the caret symbol \( \hat{1} \) indicates that we have omitted the 1 in the \( i \)-th slot so as to obtain an element of \( \mathbb{H}P^{n-1} \). Notice that we insist that the \( \xi_i^{-1} \) in this expression multiply from the right.

The general pattern for the overlap maps become clear from the example \( \varphi_3 \circ \varphi_2^{-1} \).

In the special case \( n = 1 \), we have an atlas of just two charts \( \{(U_1, \varphi_1), (U_2, \varphi_2)\} \) and in close analogy with the complex case we have \( U_1 \cap U_2 = \mathbb{H}\{0\} \) and \( \varphi_1 \circ \varphi_2^{-1}(y) = \hat{y}^{-1} = \varphi_2 \circ \varphi_1^{-1}(y) \) for \( y \in \mathbb{H}\{0\} \).

**Exercise 3.28** Show that by identifying \( \mathbb{H} \) with \( \mathbb{R}^4 \) and modifying the stereographic charts on \( S^3 \subset \mathbb{R}^4 \) we can obtain an atlas for \( S^3 \) with overlap maps of the same form as for \( \mathbb{H}P^1 \) given above. Use this to show that \( \mathbb{H}P^1 \cong S^3 \).

Combining the last exercise with previous results we have

\[ \mathbb{R}P^1 \cong S^1 \]
\[ \mathbb{C}P^1 \cong S^2 \]
\[ \mathbb{H}P^1 \cong S^3 \]

### 3.2 Homogeneous Spaces

Let \( H \) be a closed Lie subgroup of a Lie group \( G \). The we have a right action of \( H \) on \( G \) given by right multiplication \( r : G \times H \rightarrow G \). The orbits of this right action are exactly the left cosets of the quotient \( G/H \). The action is clearly free and we would like to show that it is also proper. Since we are now talking about a right action and \( G \) is the manifold on which we are acting, we need to show that the map \( P_{\text{right}} : G \times H \rightarrow G \times G \) given by \( (p, h) \mapsto (p, ph) \) is a proper map. The characterization of proper action becomes the condition that

\[ H_K := \{ h \in H : (K \cdot h) \cap K \neq \emptyset \} \]

is compact whenever \( K \) is compact. To this end let \( K \) be any compact subset of \( G \). It will suffice to show that \( H_K \) is sequentially compact and to this end let
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\{h_i\}_{i \in \mathbb{Z}_+} be a sequence in \(H_K\). Then there must be a sequences \(\{a_i\}\) and \(\{b_i\}\) in \(K\) such that \(a_i h_i = b_i\). Since \(K\) is compact and hence sequentially compact, we can pass to subsequences \(\{a_{i(j)}\}_{j \in \mathbb{Z}_+}\) and \(\{b_{i(j)}\}_{j \in \mathbb{Z}_+}\) so that \(\lim_{j \to \infty} a_{i(j)} = a\) and \(\lim_{j \to \infty} b_{i(j)} = b\). Here \(i \mapsto i(j)\) is a monotonic map on positive integers; \(\mathbb{Z}_+ \to \mathbb{Z}_+\). This means that \(\lim_{j \to \infty} h_{i(j)} = \lim_{j \to \infty} a_{i(j)}^{-1} b_{i(j)} = a^{-1} b\). Thus the original sequence \(\{h_i\}\) is shown to have a convergent subsequence and we conclude that the right action is proper. Using Theorem 3.25 (or its analogue for right actions) we obtain

**Proposition 3.29** Let \(H\) be a closed Lie subgroup of a Lie group \(G\) then

i) the right action \(G \times H \to G\) is free and proper

ii) the orbit space is the left coset space \(G/H\) and this has a unique smooth manifold structure such that the quotient map \(\pi : G \to G/H\) is a surjection. Furthermore, \(\dim(G/H) = \dim(G) - \dim(H)\).

If \(K\) is a normal Lie subgroup of \(G\) then the quotient is a group with multiplication given by \([g_1][g_2] = (g_1K)(g_2K) = g_1g_2K\). The normality of \(K\) is what makes this definition well defined. In this case, we may ask whether \(G/K\) is a Lie group. If \(K\) is closed then we know from the considerations above that \(G/K\) is a smooth manifold and that the quotient map is smooth. In fact, we have the following

**Proposition 3.30** (Quotient Lie Groups) If \(K\) is closed normal subgroup of a Lie group \(G\) then \(G/K\) is a Lie group and the quotient map \(\pi : G \to G/K\) is a Lie group homomorphism. Furthermore, if \(h : G \to H\) is a surjective Lie group homomorphism then \(\ker(h)\) is a closed normal subgroup and the induced map \(\tilde{h} : G/\ker(h) \to H\) is a Lie group isomorphism.

**Proof.** We have already observed that \(G/K\) is a smooth manifold and that the quotient map is smooth. After taking into account what we know from standard group theory the only thing we need to prove for the first part is that the multiplication and inversion in the quotient are smooth. It is an easy exercise using corollary ?? to show that both of these maps are smooth.

Consider a Lie group homomorphism \(h\) as in the hypothesis of the proposition. It is standard that \(\ker(h)\) is a normal subgroup and it is clearly closed. It is also easy to verify fact that the induced \(\tilde{h}\) map is an isomorphism. One can then use Corollary ?? to show that the induced map \(\tilde{h}\) is smooth. 

If a group \(G\) acts transitively on \(M\) (on the right or left) then \(M\) is called a **homogeneous space** with respect to that action. Of course it is possible that a single group \(G\) may act on \(M\) in more than one way and so \(M\) may be a homogeneous space in more than one way. We will give a few concrete examples shortly but we already have an abstract example on hand.

**Theorem 3.31** If \(H\) is a closed Lie subgroup of a Lie group \(G\) then the map \(G \times G/H \to G\) given by \(l : (g, g_1 H) \to g g_1 H\) is a transitive Lie group action. Thus \(G/H\) is a homogeneous space with respect to this action.
**Proof.** The fact that \( l \) is well defined follows since if \( g_1H = g_2H g_2^{-1} g_1 \in H \) and so \( gg_2H = gg_2g_2^{-1} g_1H = g_1H \). We already know that \( G/H \) is a smooth manifold and \( \pi : G \to G/H \) is a surjective submersion. We can form another submersion \( id_G \times \pi : G \times G \to G \times G/H \) making the following diagram commute:

\[
\begin{array}{ccc}
G \times G & \xrightarrow{id_G \times \pi} & G \\
\downarrow \downarrow \downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \downarrow \\
G \times G/H & \xrightarrow{\pi} & G/H
\end{array}
\]

Here the upper horizontal map is group multiplication and the lower horizontal map is the action \( l \). Since the diagonal map is smooth, it follows from Proposition ?? that \( l \) is smooth. We see that \( l \) is transitive by observing that if \( g_1H, g_2H \in G/H \) then
\[
l_{g_2g_1^{-1}}(g_1H) = g_2H
\]

It turns out that up to appropriate equivalence, the examples of the above type account for all homogeneous spaces. Before proving this let us look at some concrete examples.

**Example 3.32** Let \( M = \mathbb{R}^n \) and \( G = \text{Euc}(n, \mathbb{R}) \) the group of Euclidean motions. We realize \( \text{Euc}(n, \mathbb{R}) \) as a matrix group
\[
\text{Euc}(n, \mathbb{R}) = \left\{ \begin{bmatrix} 1 & 0 \\ v & Q \end{bmatrix} : v \in \mathbb{R}^n \text{ and } Q \in O(n) \right\}
\]

The action of \( \text{Euc}(n, \mathbb{R}) \) on \( \mathbb{R}^n \) is given by the rule
\[
\begin{bmatrix} 1 & 0 \\ v & Q \end{bmatrix} \cdot x = Qx + v
\]
where \( x \) is written as a column vector. Notice that this action is not given by a matrix multiplication but one can use the trick of representing the points \( x \) of \( \mathbb{R}^n \) by the \((n+1) \times 1\) column vectors \( \begin{bmatrix} 1 \\ v \end{bmatrix} \) and then we have
\[
\begin{bmatrix} 1 & 0 \\ v & Q \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ Qx + v \end{bmatrix}.
\]
The action is easily seen to be transitive.

**Example 3.33** As in the previous example we take \( M = \mathbb{R}^n \) but this time the group acting is the affine group \( \text{Aff}(n, \mathbb{R}) \) which we realize as a matrix group:
\[
\text{Aff}(n, \mathbb{R}) = \left\{ \begin{bmatrix} 1 & 0 \\ v & A \end{bmatrix} : v \in \mathbb{R}^n \text{ and } A \in GL(n, \mathbb{R}) \right\}
\]
The action is
\[
\begin{bmatrix} 1 & 0 \\ v & A \end{bmatrix} \cdot x = Ax + v
\]
and this is again a transitive action.
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Comparing these first two examples we see that we have made \( \mathbb{R}^n \) into a homogeneous space in two different ways. It is sometime desirable to give different names and/or notations for \( \mathbb{R}^n \) to distinguish how we are acting on the space. In the first example we might denote \( \mathbb{R}^n \) (Euclidean space) and in the second case by \( \mathbb{A}^n \) and refer to it as affine space. Note that, roughly speaking the action by \( Euc(n, \mathbb{R}) \) preserves all metric properties of figures such as curves defined in \( \mathbb{R}^n \). On the other hand, \( Aff(n, \mathbb{R}) \) always sends lines to lines, planes to planes etc.

**Example 3.34** Let \( M = H := \{ z \in \mathbb{C} : \text{Im} z > 0 \} \). This is the upper half complex plane. The group acting on \( H \) will be \( SL(2, \mathbb{R}) \) and the action is given by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.
\]

This action is transitive.

**Example 3.35** We have already seen in Example 3.7 that both \( O(n) \) and \( SO(n) \) act transitively on the sphere \( S^{n-1} \subset \mathbb{R}^n \) so \( S^{n-1} \) is a homogeneous space in at least two (slightly) different ways. Also, both \( SU(n) \) and \( U(n) \) act transitively \( S^{2n-1} \subset \mathbb{C}^n \).

**Example 3.36** Let \( V_{n,k} \) denote the set of all \( k \)-frames for \( \mathbb{R}^n \) where by a \( k \)-frame we mean an ordered set of \( k \) linearly independent vectors. Thus an \( n \)-frame is just an ordered basis for \( \mathbb{R}^n \). This set can easily be given a smooth manifold structure. This manifold is called the (real) **Stiefel manifold of \( k \)-frames**. The Lie group \( \text{GL}(n, \mathbb{R}) \) acts (smoothly) on \( V_{n,k} \) by \( g \cdot (e_1, \ldots, e_k) = (ge_1, \ldots, ge_k) \). To see that this action is transitive let \( (e_1, \ldots, e_k) \) and \( (f_1, \ldots, f_k) \) be two \( k \)-frames. Extend each to \( n \)-frames \( (e_1, \ldots, e_k, \ldots, e_n) \) and \( (f_1, \ldots, f_k, \ldots, f_n) \) since we consider elements of \( \mathbb{R}^n \) as column vectors these two \( n \)-frames can be viewed as invertible \( n \times n \) matrices \( E \) and \( F \). The if we let \( g := EF^{-1} \) then \( gE = F \) which entails \( g \cdot (e_1, \ldots, e_k) = (ge_1, \ldots, ge_k) = (f_1, \ldots, f_k) \).

**Example 3.37** Let \( V_{n,k} \) denote the set of all orthonormal \( k \)-frames for \( \mathbb{R}^n \) where by an orthonormal \( k \)-frame we mean an ordered set of \( k \) orthonormal vectors. Thus an orthonormal \( n \)-frame is just an orthonormal basis for \( \mathbb{R}^n \). This set can easily be given a smooth manifold structure and is called the **Stiefel manifold of orthonormal \( k \)-frames**. The group \( O(n, \mathbb{R}) \) acts transitively on \( V_{n,k} \) for reasons similar to those given in the last example.

**Theorem 3.38** Let \( M \) be a homogeneous space via the transitive action \( l : G \times M \to M \) and let \( G_p \) the isotropy subgroup of a point \( p \in M \). Recall that \( G \) acts on \( G/G_p \). If \( G/G_p \) is second countable (in particular if \( G \) is second countable), then there is an equivariant diffeomorphism \( \phi : G/G_p \to M \) such that \( \phi(gG_p) = g \cdot p \).

**Proof.** We want to define \( \phi \) by the rule \( \phi(gG_p) = g \cdot p \) but we must show that this is well defined. This is a standard group theory argument; if \( g_1G_p = g_2G_p \)
then \( g_1^{-1}g_2 \in G_p \) so that \((g_1^{-1}g_2) \cdot p = p\) or \(g_1 \cdot p = g_2 \cdot p\). This map is a surjective by the transitivity of the action \( l \). It is also injective since if \( \phi(g_1 G_p) = \phi(g_2 G_p) \) then \( g_1 \cdot p = g_2 \cdot p \) or \((g_1^{-1}g_2) \cdot p\) which by definition means that \( g_1^{-1}g_2 \in G_p \) and then \( g_1 G_p = g_2 G_p \). Notice that the following diagram commutes:

\[
\begin{array}{ccc}
G & \xrightarrow{\theta_p} & G/G_p \\
\downarrow & & \downarrow \\
M
\end{array}
\]

From Corollary ?? we see that \( \phi \) is smooth.

To show that \( \phi \) is a diffeomorphism it suffices to show that the rank of \( \phi \) is equal to \( \dim M \) or in other words that \( \phi \) is a submersion. Since \( \phi(g_1 G_p) = (gg_1) \cdot p = g\phi(g_1 G_p) \) the map \( \phi \) is equivariant and so has constant rank. By Lemma ?? \( \phi \) is a submersion and hence in the present case a diffeomorphism.

\[\text{Exercise 3.39} \text{ Show that if instead of the hypothesis of second countability in the last theorem we assume instead that } \theta_p \text{ has full rank at the identity then } \phi : G/G_p \to M \text{ is a diffeomorphism.}\]

Let \( l : G \times M \to M \) be a left action and fix \( p_0 \in M \). Denote the projection onto cosets by \( \pi \) and also write \( r^{p_0} : g \mapsto gp_0 \). Then we have the following equivalence of maps

\[
\begin{array}{ccc}
G & = & G \\
\pi \downarrow & & \downarrow \pi \\
G/H & \cong & M
\end{array}
\]

\[\text{Exercise 3.40} \text{ Let } G \text{ act on } M \text{ as above. Let } H_1 := G_{p_1} \text{ (isotropy of } p_1) \text{ and } H_2 := G_{p_2} \text{ (isotropy of } p_2) \text{ where } p_2 = gp_1 \text{ for some } g \in G. \text{ Show that there is a natural Lie group isomorphisms } H_1 \cong H_2 \text{ and a natural diffeomorphism } G/H_1 \cong G/H_2 \text{ which is an equivalence of actions.}\]

We now look again at some of our examples of homogeneous spaces and apply the above theorem.

\[\text{Example 3.41} \text{ Consider again Example 3.32. The isotropy group of the origin in } \mathbb{R}^n \text{ is the subgroup consisting of matrices of the form } \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \text{ where } Q \in O(n). \text{ This group is clearly isomorphic to } O(n, \mathbb{R}) \text{ and so by the above theorem we have an equivariant diffeomorphism } \mathbb{R}^n \cong Euc(n, \mathbb{R})/O(n, \mathbb{R})\]

Example 3.42  Consider again Example 3.33. The isotropy group of the origin in $\mathbb{R}^n$ is the subgroup consisting of matrices of the form
\[
\begin{pmatrix}
1 & 0 \\
0 & A
\end{pmatrix}
\]
where $A \in \text{Gl}(n, \mathbb{R})$. This group is clearly isomorphic to $\text{Gl}(n, \mathbb{R})$ and so by the above theorem we have an equivariant diffeomorphism
\[
\mathbb{R}^n \cong \frac{\text{Aff}(n, \mathbb{R})}{\text{Gl}(n, \mathbb{R})}
\]
It is important to realize that there is an implied action on $\mathbb{R}^n$ which is different from that in the previous example.

Example 3.43  Now consider the action of $\text{Sl}(2, \mathbb{R})$ on the upper half complex plane as in Example 3.34. Let us determine the isotropy subgroup for the point $i = \sqrt{-1}$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in this subgroup then
\[
ai + b = i
\]
so that $bc - ad = 1$ and $bd + ac = 0$. Thus $A \in \text{SO}(2, \mathbb{R})$ (which is isomorphic as a Lie group to the circle $S^1 = U(1, \mathbb{C})$). Thus we have an equivariant diffeomorphism
\[
H = \mathbb{C}_+ \cong \frac{\text{Sl}(2, \mathbb{R})}{\text{SO}(2, \mathbb{R})}
\]

Example 3.44  From example 3.35 we obtain
\[
\begin{align*}
S^{n-1} & \cong \frac{\text{O}(n)}{\text{O}(n-1)} \\
S^{n-1} & \cong \frac{\text{SO}(n)}{\text{SO}(n-1)} \\
S^{2n-1} & \cong \frac{\text{U}(n)}{\text{U}(n-1)} \\
S^{2n-1} & \cong \frac{\text{SU}(n)}{\text{SU}(n-1)}
\end{align*}
\]

Example 3.45  Let $(e_1, ..., e_n)$ be the standard basis for $\mathbb{R}^n$. Under the action of $\text{Gl}(n, \mathbb{R})$ on $V'_{n,k}$ given in Example 3.36, the isotropy group of the point which is the $k$-plane $e = (e_{k+1}, ..., e_n)$ is the subgroup of $\text{Gl}(n, \mathbb{R})$ of the form
\[
\begin{pmatrix}
A & 0 \\
0 & \text{id}
\end{pmatrix}
\]
for $A \in \text{Gl}(n - k, \mathbb{R})$

We identify this group with $\text{Gl}(n - k, \mathbb{R})$ and then we obtain
\[
V'_{n,k} \cong \frac{\text{Gl}(n, \mathbb{R})}{\text{Gl}(n - k, \mathbb{R})}
\]
Example 3.46 A similar analysis leads to an equivariant diffeomorphism

\[ V_{n,k} \cong \frac{O(n, \mathbb{R})}{O(n-k, \mathbb{R})} \]

where \( V_{n,k} \) is the Stiefel manifold of orthonormal \( k \)-planes of Example 3.37. Notice that taking \( k = 1 \) we recover Example 3.41.

Exercise 3.47 Show that if \( k < n \) then we have

\[ V_{n,k} \cong \frac{SO(n, \mathbb{R})}{SO(n-k, \mathbb{R})} \]

Next we introduce a couple of standard results concerning connectivity.

Proposition 3.48 Let \( G \) be a Lie group acting freely and properly on a smooth manifold \( M \). Let the action be a left (resp. right) action. If both \( G \) and \( M\setminus G \) (resp. \( M/G \)) are connected then \( M \) is connected.

Proof. Assume for concreteness that the action is a left action and that \( G \) and \( M\setminus G \) are connected. Suppose by way of contradiction that \( M \) is not connected. Then there are disjoint open set \( U \) and \( V \) whose union is \( M \). Each orbit \( G \cdot p \) is the image of the connected space \( G \) under the orbit map \( g \mapsto g \cdot p \) and so is connected. This means that each orbit must be contained in one and only one of \( U \) and \( V \). Now since the quotient map \( \pi \) is an open map, \( \pi(U) \) and \( \pi(V) \) are open and from what we have just observed they must be disjoint and \( \pi(U) \cup \pi(V) = M\setminus G \). This contradicts the assumption that \( M\setminus G \) is connected.

Corollary 3.49 Let \( H \) be a closed Lie subgroup of \( G \). Then if both \( H \) and \( G/H \) are connected then \( G \) is connected.

Corollary 3.50 For each \( n \geq 1 \) then groups \( SO(n) \), \( SU(n) \) and \( U(n) \) are connected while the group \( O(n) \) has exactly two components.

Proof. \( SO(1) \) and \( SU(1) \) are both connected since they each contain only one element. \( U(1) \) is the circle and so it too is connected. We use induction. Suppose that \( SO(k) \), \( SU(k) \) are connected for \( 1 \leq k \leq n-1 \). We show that this implies that \( SO(n) \), \( SU(n) \) and \( U(n) \) are connected. From example 3.44 we know that \( S^{n-1} = SO(n)/SO(n-1) \). Since \( S^{n-1} \) and \( SO(n-1) \) are connected (the second one by the induction hypothesis) we see that \( SO(n) \) is connected. The same argument works for \( SU(n) \) and \( U(n) \).

Every element of \( O(n) \) has either determinant 1 or \(-1\). The subset \( SO(n) \subset O(n) \) since it is exactly \( \{ g \in O(n) : \det g \neq 1 \} \). If we fix an element \( a_0 \) with \( \det a_0 = -1 \) then \( SO(n) \) and \( a_0SO(n) \) are disjoint, open and both connected since \( g \mapsto a_0g \) is a diffeomorphism which maps the first to the second. It is easy to show that \( SO(n) \cup a_0SO(n) = O(n) \).

We close this chapter by relating the notion of a Lie group action with that of a Lie group representation. We given just a few basic definitions, some of which will be used in the next chapter.
3.2. HOMOGENEOUS SPACES

Definition 3.51 A representation of a Lie group $G$ in a finite dimensional vector space $V$ is a left Lie group action $\lambda : G \times V \to V$ such that for each $g \in G$ the map $\lambda_g : v \mapsto \lambda(g, v)$ is linear. Thus a representation is a linear action.

The map $G \to \text{Gl}(V)$ given by $g \mapsto \lambda(g)$ is a Lie group homomorphism and will be denoted by the same letter $\lambda$ as the action so that $\lambda(g)v := \lambda(g, v)$. In fact, given a Lie group homomorphism $\lambda : G \to \text{Gl}(V)$ we obtain a linear action by letting $\lambda(g, v) := \lambda(g)v$. Thus a representation is basically the same thing as a Lie group homomorphism into $\text{Gl}(V)$ and is often defined as such. The kernel of the action is the kernel of the associated homomorphism. A **faithful representation** is one that acts effectively and this means that the associated homomorphism has trivial kernel.

Exercise 3.52 Show that if $\lambda : G \times V \to V$ is a map such that $\lambda_g : v \mapsto \lambda(g, v)$ is linear for all $g$, then $\lambda$ is smooth if and only if $\lambda_g : G \to \text{Gl}(V)$ is smooth for every $g \in G$. (Assume $V$ is finite dimensional as usual).

We have already seen one important example of a Lie group representation. Namely, the adjoint representation. The adjoint representation came from first considering the action of $G$ on itself given by conjugation which leaves the identity element fixed. The idea can be generalized:

Theorem 3.53 Let $l : G \times M \to M$ be a (left) Lie group action. Suppose that $p_0 \in M$ is a fixed point of the action ($l_g(p_0) = p_0$ for all $g$). The map $l^{(p_0)} : G \to \text{Gl}(T_{p_0}M)$ given by $l^{(p_0)}(g) := T_{p_0}l_g$ is a Lie group representation.

**Proof.** Since

$$l^{(p_0)}(g_1g_2) = T_{p_0}(l_{g_1g_2}) = T_{p_0}(l_{g_1} \circ l_{g_2}) = T_{p_0}l_{g_1} \circ T_{p_0}l_{g_2} = l^{(p_0)}(g_1)l^{(p_0)}(g_2)$$

we see that $l^{(p_0)}$ is a homomorphism. We must show that $l^{(p_0)}$ is smooth. By Exercise 3.52 this implies that the map $G \times T_{p_0}M \to T_{p_0}M$ given by $(g, v) \mapsto T_{p_0}l_g \cdot v$ is smooth. It will be enough to show that $g \mapsto \alpha(T_{p_0}l_g \cdot v)$ is smooth for any $v \in T_{p_0}M$ and any $\alpha \in T_{p_0}^*M$. This will follow if we can show that for fixed $v_0 \in T_{p_0}M$, the map $G \to TM$ given by $g \mapsto T_{p_0}l_g \cdot v_0$ is smooth. This map is a composition

$$G \to TG \times TM \cong T(G \times M) \xrightarrow{Tl} TM$$

where the first map is $g \mapsto (0_g, v_0)$ which is clearly smooth. ■
CHAPTER 3. LIE GROUPS II

Definition 3.54 For a Lie group action $l : G \times M \to M$ with fixed point $p_0$ the representation $l(p_0)$ from the last theorem is called the isotropy representation for the fixed point.

Now let us consider a transitive Lie group action $l : G \times M \to M$ and a point $p_0$. For notational convenience, denote the isotropy subgroup $G_{p_0}$ by $H$. Then $H$ acts on $M$ by restriction. We denote this action by $\lambda : H \times M \to M$

$$\lambda : (h, p) \mapsto h$$

for $h \in H = G_{p_0}$.

Now notice that $p_0$ is a fixed point of this action and so we have an isotropy representation $\lambda_{p_0} : T_{p_0} M \to T_{p_0} M$. On the other hand, we have another action $C : H \times G \to G$ where $C_h : G \to G$ is given by $g \mapsto hgh^{-1}$ for $h \in H$. The Lie differential of $C_h$ is the adjoint map $\text{Ad}_h : g \mapsto g$. The map $C_h$ fixes $H$ and $\text{Ad}_h$ fixes $\mathfrak{h}$. Thus the map $\text{Ad}_h : g \mapsto g$ descends to a map $\tilde{\text{Ad}}_h : g/\mathfrak{h} \to g/\mathfrak{h}$. We are going to show that there is a natural isomorphism $T_{p_0} M \cong g/\mathfrak{h}$ such that for each $h \in H$ the following diagram commutes:

$$\tilde{\text{Ad}}_h : g/\mathfrak{h} \to g/\mathfrak{h}$$

$$\downarrow \quad \downarrow$$

$$\lambda_{p_0} : T_{p_0} M \to T_{p_0} M$$

One way to state the meaning of this result is to say that $h \mapsto \tilde{\text{Ad}}_h$ is a representation of $H$ on the vector space $g/\mathfrak{h}$ which is equivalent to the linear isotropy representation. The isomorphism $T_{p_0} M \cong g/\mathfrak{h}$ is given in the following very natural way: Let $\xi \in \mathfrak{h}$ and consider $T_e \pi(\xi) \in T_{p_0} M$. We have

$$T_e \pi(\xi) = \left. \frac{d}{dt} \right|_{t=0} \pi(\exp(\xi t)) = 0$$

since $\exp(\xi t) \in \mathfrak{h}$ for all $t$. Thus $\mathfrak{h} \subset \ker(T_e \pi)$. On the other hand, $\dim \mathfrak{h} = \dim H = \dim(\ker(T_e \pi))$ so in fact $\mathfrak{h} = \ker(T_e \pi)$ and we obtain an isomorphism $g/\mathfrak{h} \cong T_{p_0} M$ induced $T_e \pi$. Let us now see why the diagram 3.1 commutes. Let us take a scenic root to the conclusion since it allows us to see the big picture a bit better. First the following diagram clearly commutes:

$$G \xrightarrow{C_h} G$$

$$\pi \downarrow \quad \pi \downarrow$$

$$G/H \xrightarrow{L_h} G/H$$

Elementwise we have

$$g \xrightarrow{C_h} hgh^{-1}$$

$$\pi \downarrow \quad \pi \downarrow$$

$$gH \xrightarrow{L_h} hH$$

Using our equivariant diffeomorphism $\phi : G/H \to M$ we obtain an equivalent commutative diagram:

$$G \xrightarrow{C_h} G$$

$$r^{p_0} \downarrow \quad r^{p_0} \downarrow$$

$$M \xrightarrow{\lambda_{p_0}} M$$
3.2. HOMOGENEOUS SPACES

Applying these maps to $\exp t\xi$ for $\xi \in \mathfrak{g}$ we have

$$
\begin{align*}
\exp t\xi & \rightarrow h(\exp t\xi)h^{-1} \\
_{r^0p_0} \downarrow & \downarrow_{r^0p_0} \\
(\exp t\xi)p_0 & \rightarrow h(\exp t\xi)p_0
\end{align*}
$$

Applying the tangent functor (looking at the differential) we get the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\text{Ad}_h} & \mathfrak{g} \\
T_{r^0p_0} & \downarrow & T_{r^0p_0} \\
T_{p_0M} & \xrightarrow{\lambda_{(p_0)}^h} & T_{p_0M}
\end{array}
$$

and taking quotients, this gives the desired commutative diagram 3.1.

### 3.2.1 Reductive case

**Definition 3.55** If $M$ is a homogeneous space with group $G$ and $H = G_{p_0}$, then we say that it is a **reductive** homogeneous space if there exist a subspace $\mathfrak{m} \subset \mathfrak{g}$ which is complementary to $\mathfrak{h}$ such that $\text{Ad}_h(\mathfrak{m}) \subset \mathfrak{m}$ for all $h \in H$.

In general, we do not claim that $\mathfrak{m}$ is a Lie subalgebra of $\mathfrak{g}$. The condition on $\mathfrak{m}$ is that it be $\text{Ad}(H)$-invariant. Since $\mathfrak{m}$ is complementary to $\mathfrak{h}$ we have $\mathfrak{m} \cong \mathfrak{g}/\mathfrak{h}$. Thus $\mathfrak{m} \cong T_{p_0M}$. However, the assumption that $\mathfrak{m}$ is $\text{Ad}(H)$-invariant gives more. Namely, $h \mapsto \text{Ad}_h|_{\mathfrak{m}}$ is a representation of $H$ in $\mathfrak{m}$ and it is equivalent to the isotropy representation. For the reductive case, we have a commutative diagram valid for all $h \in H$:

$$
\begin{array}{ccc}
\mathfrak{m} & \xrightarrow{\text{Ad}_h|_{\mathfrak{m}}} & \mathfrak{m} \\
\downarrow & \downarrow & \downarrow \\
\widetilde{\text{Ad}}_h : \mathfrak{g}/\mathfrak{h} & \rightarrow & \mathfrak{g}/\mathfrak{h} \\
\downarrow & \downarrow & \downarrow \\
\lambda_{(p_0)}^h : T_{p_0M} & \rightarrow & T_{p_0M}
\end{array}
$$

where the map $\mathfrak{m} \rightarrow \mathfrak{g}/\mathfrak{h}$ is the restriction of the quotient map $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$ to the subspace $\mathfrak{m}$. It is a linear isomorphism. In the reductive case, each of the maps $h \mapsto \text{Ad}_h|_{\mathfrak{m}}$, $h \mapsto \widetilde{\text{Ad}}_h$ and $h \mapsto \lambda_{(p_0)}^h$ are equivalent and can be considered as a version the isotropy representation.
Suppose that $V$ is an $F$-vector space and let $B = \{v_1, \ldots, v_n\}$ be a basis for $V$. Then denoting the matrix representative of $\lambda_g$ with respect to $B$ by $[\lambda_g]_B$ we obtain a homomorphism $G \rightarrow \text{GL}(n, F)$ given by $g \mapsto [\lambda_g]_B$. In general, a Lie group homomorphism of a Lie group $G$ into $\text{GL}(n, F)$ is called a matrix representation of $G$. Notice that any Lie subgroup $G$ of $\text{GL}(V)$ acts on $V$ in the obvious way simply by employing the definition of $\text{GL}(V)$ as a set of linear transformations of $V$. We call this the standard action of the linear Lie subgroup of $\text{GL}(V)$ on $V$ and the corresponding homomorphism is just the inclusion map $G \hookrightarrow \text{GL}(n, F)$. Choosing a basis, the subgroup corresponds to a matrix group and the standard action becomes matrix multiplication on the left of $F^n$ where the latter is viewed as a space of column vectors. This action of a matrix group on column vectors is also referred to as a standard action.

Given a representation $\lambda$ of $G$ in a vector space $V$ we have a representation $\lambda^*$ of $G$ in the dual space $V^*$ by defining $\lambda^*(g) := (\lambda(g^{-1}))^t : V^* \rightarrow V^*$. Here we have by definition $\langle (\lambda(g^{-1}))^t v, w \rangle = \langle \lambda(g^{-1})v, w \rangle$ for all $v, w \in V$ and where $\langle \langle \ldots \rangle : V^* \times V \rightarrow F$ is the natural bilinear pairing which defines the dual. This dual representation is also sometimes called the contragradient representation (especially when $F = \mathbb{R}$).

Now let $\lambda^V$ and $\lambda^W$ be representations of a lie group $G$ in $F$-vector spaces $V$ and $W$ respectively. We can then form the direct product representation $\lambda^V \oplus \lambda^W$ by $(\lambda^V \oplus \lambda^W)_g := \lambda^V_g \oplus \lambda^W_g$ for $g \in G$ and where we have $(\lambda^V_g \oplus \lambda^W_g)(v, w) = (\lambda^V_g v, \lambda^W_g w)$.

One can also form the tensor product of representations. The definitions and basic facts about tensor products are given in the more general context of module theory in Appendix ???. Here we given a quick recounting of the notion of a tensor product of vector spaces and then we defined tensor products of representations. Given two vector spaces $V_1$ and $V_2$ over some field $F$. Consider
the space $\mathcal{C}_{V_1 \times V_2}$ consisting of all bilinear maps $V_1 \times V_2 \to W$ where $W$ varies over all $F$-vector spaces but $V_1$ and $V_2$ are fixed. A morphism from, say $\mu_1 : V_1 \times V_2 \to W_1$ to $\mu_2 : V_1 \times V_2 \to W_2$ is defined to be a map $\ell : W_1 \to W_2$ such that the diagram

\[
\begin{array}{ccc}
W_1 & \xrightarrow{\mu_1} & W_2 \\
\downarrow & & \downarrow \\
V_1 \times V_2 & \xrightarrow{\ell} & W \\
\uparrow & & \uparrow \\
W_2 & \xrightarrow{\mu_2} & W_1
\end{array}
\]

commutes.

There exists a vector space $T_{V_1, V_2}$ together with a bilinear map $\otimes : V_1 \times V_2 \to T_{V_1, V_2}$ that has the following universal property: For every bilinear map $\mu : V_1 \times V_2 \to W$ there is a unique linear map $\tilde{\mu} : T_{V_1, V_2} \to W$ such that the following diagram commutes:

\[
\begin{array}{ccc}
V_1 \times V_2 & \xrightarrow{\mu} & W \\
\downarrow & & \downarrow \\
T_{V_1, V_2} & \xrightarrow{\tilde{\mu}} & W \\
\downarrow & & \downarrow \\
& & \\
& & \\
\end{array}
\]

If such a pair $(T_{V_1, V_2}, \otimes)$ exists with this property then it is unique up to isomorphism in $\mathcal{C}_{V_1 \times V_2}$. In other words, if $\tilde{\otimes} : V_1 \times V_2 \to \tilde{T}_{V_1, V_2}$ is another object with this universal property then there is a linear isomorphism $T_{V_1, V_2} \cong \tilde{T}_{V_1, V_2}$ such that the following diagram commutes:

\[
\begin{array}{ccc}
V_1 \times V_2 & \xrightarrow{\otimes} & T_{V_1, V_2} \\
\downarrow & \cong & \downarrow \\
\tilde{V}_1 \times \tilde{V}_2 & \xrightarrow{\tilde{\otimes}} & \tilde{T}_{V_1, V_2}
\end{array}
\]

We refer to any such universal object as a tensor product of $V_1$ and $V_2$. We will indicate the construction of a specific tensor product that we denote by $V_1 \otimes V_2$ with corresponding map $\otimes : V_1 \times V_2 \to V_1 \otimes V_2$. The idea is simple: We let $V_1 \otimes V_2$ be the set of all linear combinations of symbols of the form

\[
\begin{align*}
&v_1 \otimes v_2, \\
&v_1' \otimes v_2' \otimes \ldots \\
&\vdots \\
&\ldots \otimes v_n' \otimes v_n
\end{align*}
\]
$v_1 \otimes v_2$ for $v_1 \in V_1$ and $v_2 \in V_2$, subject to the relations

\[
(v_1 + v_2) \otimes v = v_1 \otimes v + v_2 \otimes v \\
 v \otimes (v_1 + v_2) = v \otimes v_1 + v \otimes v_2 \\
r(v_1 \otimes v_2) = rv_1 \otimes v_2 = v_1 \otimes rv_2
\]

The map $\otimes$ is then simply $\otimes : (v_1, v_2) \rightarrow v_1 \otimes v_2$. Somewhat more pedantically, let $F(V_1 \times V_2)$ denote the free vector space generated by the set $V_1 \times V_2$ (the elements of $V_1 \times V_2$ are treated as a basis for the space and so the free space has dimension equal to the cardinality of the set $V_1 \times V_2$). Next we define an equivalence relation \( \sim \) $F(V_1 \times V_2)$ generated by the relations

\[
(av_1, v_2) \sim a(v_1, v_2) \\
(v_1, av_2) \sim a(v_1, v_2)
\]

\[
(v + w, v_2) \sim (v, v_2) + (w, v_2) \\
(v_1, v + w) \sim (v_1, v) + (v_1, w)
\]

for $v_1, v \in V_1$, $v_2, w \in V_2$ and $a \in F$. Then we let $V_1 \otimes V_2 := F(V_1 \times V_2) / \sim$ and denote the equivalence class of $(v_1, v_2)$ by $v_1 \otimes v_2$.

Tensor products of several vector spaces at a time are constructed similarly to be a universal space in a category of multilinear maps. We may also form the tensor products of several vector spaces at a time are constructed similarly to be a universal space in a category of multilinear maps. We may also form the tensor products two at a time and then use the easily proved fact

\[
V \otimes W = V \otimes (W \otimes U) \cong V \otimes W \otimes U
\]

which is then denoted by $V \otimes W \otimes U$. Again the reader is referred to Appendix ?? for more about tensor products.

Elements of the form $v_1 \otimes v_2$ generate $V_1 \otimes V_2$ and in fact, if \{e$_1$, ..., e$_r$\} is a basis for $V_1$ and \{f$_1$, ..., f$_s$\} is a basis for $V_2$ then set

\[
\{e_i \otimes f_j : 1 \leq i \leq r, 1 \leq j \leq s\}
\]

is a basis for $V_1 \otimes V_2$ which therefore has dimension $rs = \dim V_1 \dim V_2$.

One more observation: If $A : V_1 \rightarrow W_1$ and $B : V_2 \rightarrow W_2$ are linear, then we can define a linear map $A \otimes B : V_1 \otimes V_2 \rightarrow W_1 \otimes W_2$. It is enough to define $A \otimes B$ on elements of the form $v_1 \otimes v_2$ and then extend linearly:

\[
A \otimes B(v_1 \otimes v_2) = Av_1 \otimes Bv_2
\]

Notice that if $A$ and $B$ are invertible then $A \otimes B$ is invertible with $(A \otimes B)^{(v_1 \otimes v_2)} = A^{-1}v_1 \otimes B^{-1}v_2$. Pick bases for $V_1$ and $V_2$ as above and bases \{e$_1$, ..., e'$_r$\} and \{f$_1$, ..., f'$_s$\} for $W_1$ and $W_2$ respectively. The for $\tau \in V_1 \otimes V_2$ we can write $\tau = \tau^{ij}e_i \otimes f_j$ using the Einstein summation convention. We have

\[
A \otimes B(\tau) = A \otimes B(\tau^{ij}e_i \otimes f_j)
\]

\[
= \tau^{ij}A_{ik} \otimes B_{lj}f_j
\]

\[
= \tau^{ij}A_{ik}^{'}e_{'k} \otimes B_{lj}'f_j'
\]

\[
= \tau^{ij}A_{ik}^{'}B_{lj}' (e_{'k} \otimes f_{'j})
\]
so that the matrix of $A \otimes B$ is given by $(A \otimes B)^{kl}_{ij} = A^k_i B^l_j$.

Now let $\lambda^V$ and $\lambda^W$ be representations of a lie group $G$ in $F$-vector spaces $V$ and $W$ respectively. We can form a representation of $G$ in the tensor product space by letting $\left(\lambda^V \otimes \lambda^W\right)_g := \lambda^V_g \otimes \lambda^W_g$ for all $g \in G$. There is a variation on the tensor product that is useful when we have two groups involved. If $\lambda^V$ is a representation of a lie group $G_1$ in $F$-vector space $V$ and $\lambda^W$ is representation of a lie group $G_2$ in the $F$-vector space $W$, then we can form a representation of the Lie group $G_1 \times G_2$ also called the tensor product representation and denoted $\lambda^V \otimes \lambda^W$ as before. In this case the definition is $\left(\lambda^V \otimes \lambda^W\right)_{(g_1, g_2)} := \lambda^V_{g_1} \otimes \lambda^W_{g_2}$. Of course if it happens that $G_1 = G_2$ then have an ambiguity since $\lambda^V \otimes \lambda^W$ could be a representation of $G$ or of $G \times G$. One usually determines which version is meant from the context. Alternatively, one can use pairs to denote actions so that an action $\lambda : G \times V \to V$ is denoted $(G, \lambda)$. Then the two tensor product representations would be $(G \times G, \lambda^V \otimes \lambda^W)$ and $(G, \lambda^V \otimes \lambda^W)$ respectively.
Chapter 4

Killing Fields and Symmetric Spaces

Following the excellent treatment given by Lang [?], we will treat Killing fields in a more general setting than is traditional. We focus attention on pairs \((M, \nabla)\) where \(M\) is a smooth manifold and \(\nabla\) is a (not necessarily metric) torsion free connection. Now let \((M, \nabla^M)\) and \((N, \nabla^N)\) be two such pairs. For a diffeomorphism \(\varphi : M \to N\) the pull-back connection \(\varphi^\ast \nabla\) is defined so that

\[(\varphi^\ast \nabla^N)\varphi^\ast X \varphi^\ast Y = \varphi^\ast (\nabla^M_X Y)\]

for \(X, Y \in \mathfrak{X}(N)\) and \(\varphi^\ast X\) is defined as before by \(\varphi^\ast X = T\varphi^{-1} \circ X \circ \varphi\). An isomorphism of pairs \((M, \nabla^M) \to (N, \nabla^N)\) is a diffeomorphism \(\varphi : M \to M\) such that \(\varphi^\ast \nabla = \nabla\). Automorphism of a pair \((M, \nabla)\) is defined in the obvious way. Because we will be dealing with the flows of vector fields which are not necessarily complete we will have to deal with maps \(\varphi : M \to M\) with domain a proper open subset \(O \subset M\). In this case we say that \(\varphi\) is a local isomorphism if \(\varphi^\ast \nabla|_{\varphi^{-1}(O)} = \nabla|_{O}\).

**Definition 4.1** A vector field \(X \in \mathfrak{X}(M)\) is called a \(\nabla\)-**Killing field** if \(\varphi^t X\) is a local isomorphism of the pair \((M, \nabla)\) for all sufficiently small \(t\).

**Definition 4.2** Let \(M, g\) be a semi-Riemannian manifold. A vector field \(X \in \mathfrak{X}(M)\) is called a \(g\)-**Killing field** if \(X\) is a local isometry.

It is clear that if \(\nabla^g\) is the Levi-Civita connection for \(M, g\) then any isometry of \(M, g\) is an automorphism of \((M, \nabla^g)\) and any \(g\)-Killing field is a \(\nabla^g\)-Killing field. The reverse statements are not always true. Letting \(\text{Kill}_g(M)\) denote the \(g\)-Killing fields and \(\text{Kill}_{\nabla^g}(M)\) the \(\nabla^g\)-Killing fields we have

\[\text{Kill}_g(M) \subset \text{Kill}_{\nabla^g}(M)\]
**Lemma 4.3** If $\langle M, \nabla \rangle$ is as above then for any $X, Y, Z \in \mathfrak{X}(M)$ we have

$$[X, \nabla_Z Y] = \nabla_{Y,Z} - R_{Y,Z}X + \nabla_{[X,Z]}Y + \nabla_Z [X,Y]$$

**Proof.** The proof is a straightforward calculation left to the reader. ■

**Theorem 4.4** Let $M$ be a smooth manifold and $\nabla$ a torsion free connection. For $X \in \mathfrak{X}(M)$, the following three conditions are equivalent:

(i) $X$ is a $\nabla$–Killing field

(ii) $[X, \nabla_Z Y] = \nabla_{[X,Z]}Y + \nabla_Z [X,Y]$ for all $Y, Z \in \mathfrak{X}(M)$

(iii) $\nabla_{Y,Z}X = R_{Y,Z}X$ for all $Y, Z \in \mathfrak{X}(M)$.

**Proof.** The equivalence of (ii) and (iii) follows from the previous lemma.

Let $\phi_t := \varphi_t^X$. If $X$ is Killing (so (i) is true) then locally we have $\frac{d}{dt}\varphi_t^X Y = \varphi_t^X \mathcal{L}_X Y = \varphi_t^X [X,Y]$ for all $Y \in \mathfrak{X}(M)$. We also have $\phi_t^X X = X$. One calculates that

$$\frac{d}{dt} \phi_t^* (\nabla_Z Y) = \phi_t^* [X, \nabla_Z Y] = [\phi_t^* X, \phi_t^* \nabla_Z Y]$$

and on the other hand

$$\nabla_{\phi_t^*Z} \phi_t^* Y = \nabla_{\phi_t^*[X,Z]} \phi_t^* Y + \nabla_{\phi_t^*Z} (\phi_t^*[X,Y])$$

Setting $t = 0$ and comparing we get (ii).

Now assume (ii). We would like to show that $\phi_{-t}^* \nabla_{\phi_t^*Z} \phi_t^* Y = \nabla_Z Y$. We show that $\frac{d}{dt}\phi_{-t}^* \nabla_{\phi_t^*Z} \phi_t^* Y = 0$ for all sufficiently small $t$. The thing to notice here is that since the difference of connections is tensorial $\tau(Y, Z) = \frac{d}{dt}\phi_{-t}^* \nabla_{\phi_t^*Z} \phi_t^* Y$ is tensorial being the limit of the a difference quotient. Thus we can assume that $[X,Y] = [X,Z] = 0$. Thus $\phi_t^* Z = Z$ and $\phi_t^* Y = Y$. We now have

$$\frac{d}{dt} \phi_{-t}^* \nabla_{\phi_t^*Z} \phi_t^* Y$$

and

$$\frac{d}{dt} \phi_{-t}^* (\nabla_Z Y) = \phi_{-t}^* [X, \nabla_Z Y]$$

and

$$\phi_{-t}^* (\nabla_{[X,Z]}Y + \nabla_Z [X,Y]) = 0$$

But since $\phi_{-t}^* \nabla_{\phi_t^*Z} \phi_t^* Y$ is equal to $\nabla_Z Y$ when $t = 0$, we have that $\phi_{-t}^* \nabla_{\phi_t^*Z} \phi_t^* Y = \nabla_Z Y$ for all $t$ which is (i). ■

Clearly the notion of a Jacobi field makes sense in the context of a general (torsion free) connection on $M$. Notice also that an automorphism $\phi$ of a pair $(M, \nabla)$ has the property that $\gamma \circ \phi$ is a geodesic if and only if $\phi$ is a geodesic.

**Proposition 4.5** $X$ is a $\nabla$–Killing field if and only if $X \circ \gamma$ is a Jacobi field along $\gamma$ for every geodesic $\gamma$. 

Proof. If $X$ is Killing then $(s, t) \mapsto \varphi_t^X(\gamma(t))$ is a variation of $\gamma$ through geodesics and so $t \mapsto X \circ \gamma(t) = \frac{d}{dt} \big|_{t=0} \varphi_t^X(\gamma(t))$ is a Jacobi field. The proof of the converse (Karcher) is as follows: Suppose that $X$ is such that its restriction to any geodesic is a Jacobi field. Then for $\gamma$ a geodesic, we have

$$\nabla^2_\gamma (X \circ \gamma) = R(\dot{\gamma}, X \circ \gamma) \dot{\gamma}$$

where we have used $\dot{\gamma}$ to denote not only a field along $\gamma$ but also an extension to a neighborhood with $[\dot{\gamma}, X] = 0$. But

$$\nabla_\gamma \nabla \dot{\gamma} = \nabla_\gamma \nabla_\gamma X - \nabla_\gamma [\dot{\gamma}, X] = \nabla_\gamma \nabla_\gamma X$$

and so $\nabla^2_\gamma (X \circ \gamma) = \nabla \gamma \nabla \dot{\gamma} X$. Now let $v, w \in T_pM$ for $p \in M$. Then there is a geodesic $\gamma$ with $\dot{\gamma}(0) = v + w$ and so

$$R(v, X)w + R(w, X)v + R(v, X)v + R(w, X)v = R(v + w, X)(v + w) = \nabla \gamma \nabla \dot{\gamma} X = \nabla_{v+w,v+w} X$$

$$= \nabla_{v,v} X + \nabla_{v,v} X + \nabla_{w,w} X + \nabla_{w,w} X + \nabla_{v,v} X$$

Now replace $w$ with $-w$ and subtract (polarization) to get

$$\nabla_{v,v} X + \nabla_{v,v} X = R(v, X)w + R(w, X)v.$$  

On the other hand, $\nabla_{v,v} X - \nabla_{w,w} X = R(v, w)X$ and adding this equation to the previous one we get $2\nabla_{v,w} X = R(v, X)w - R(X, w)v - R(w, v)X$ and then by the Bianchi identity $2\nabla_{v,w} X = 2R(v, w)X$. The result now follows from theorem 4.4.

We now give equivalent conditions for $X \in \mathfrak{X}(M)$ to be a $g$–Killing field for a semi-Riemannian $(M, g)$. First we need a lemma.

**Lemma 4.6** For any vector fields $X, Y, Z \in \mathfrak{X}(M)$ we have

$$\mathcal{L}_X (Y, Z) = \langle \mathcal{L}_X Y, Z \rangle + \langle Y, \mathcal{L}_X Z \rangle + \langle \mathcal{L}_X g \rangle (Y, Z)$$

$$= \langle \mathcal{L}_X Y, Z \rangle + \langle Y, \mathcal{L}_X Z \rangle + \langle \mathcal{L}_X Y, Z \rangle + \langle Y, \nabla X Z \rangle$$

**Proof.** This is again a straightforward calculation using for the second equality

$$(\mathcal{L}_X g) (Y, Z) = \mathcal{L}_X (Y, Z) - \langle \mathcal{L}_X Y, Z \rangle - \langle Y, \mathcal{L}_X Z \rangle$$

$$= \mathcal{L}_X (Y, Z) + \langle \nabla X Y - \nabla Y X, Z \rangle + \langle Y, \nabla X Z - \nabla Z X \rangle$$

$$= \mathcal{L}_X (Y, Z) + \langle \nabla X Y - \nabla Y X, Z \rangle + \langle Y, \nabla X Z - \nabla Z X \rangle$$

$$= \mathcal{L}_X (Y, Z) - \langle \nabla X Y, Z \rangle - \langle Y, \nabla Z X \rangle + \langle \nabla X Y, Z \rangle + \langle Y, \nabla X Z \rangle$$

$$0 + \langle \nabla X Y, Z \rangle + \langle Y, \nabla X Z \rangle.$$
Theorem 4.7 Let $(M,g)$ be semi-Riemannian. $X \in \mathfrak{X}(M)$ is a $g$–Killing field if and only if any one of the following conditions hold:

(i) $\mathcal{L}_X g = 0$.

(ii) $\mathcal{L}_X (Y, Z) = \langle \mathcal{L}_X Y, Z \rangle + \langle Y, \mathcal{L}_X Z \rangle$ for all $Y, Z \in \mathfrak{X}(M)$.

(iii) For each $p \in M$, the map $T_p M \to T_p M$ given by $v \mapsto -\nabla_v X$ is skew-symmetric for the inner product $\langle \cdot, \cdot \rangle_p$.

Proof. Since if $X$ is Killing then $\phi_t^X g = g$ and in general for a vector field $X$ we have

$$\frac{d}{dt} \phi_t^X g = \phi_t^X \mathcal{L}_X g$$

the equivalence of (i) with the statement that $X$ is $g$–Killing is immediate.

If (ii) holds then by the previous lemma, (i) holds and conversely. The equivalence of (ii) and (i) also follows from the lemma and is left to the reader.
Chapter 5

Fiber Bundles II

5.1 Principal and Associated Bundles

Let \( \pi : E \to M \) be a vector bundle with typical fiber \( V \) and for every \( p \in M \) let \( GL(V, E_p) \) denote the set of linear isomorphisms from \( V \) to \( E_p \). If we choose a fixed basis \( \{e_i\}_{i=1}^k \) for \( V \) then we can identify each frame \( u = (u_1, \ldots, u_k) \) at \( p \) with the element of \( GL(V, E_p) \) given by \( u(v) := \sum v^i u_i \) where \( v = \sum v^i e_i \). With this identification, notice that if \( \sigma_\alpha := \sigma_{\phi_\alpha} \) is the local section coming from a VB-chart \( (U_\alpha, \phi_\alpha) \) as described above then we have

\[ \sigma_\alpha(p) = \Phi_\alpha|_{E_p} \quad \text{for } p \in U_\alpha \]

Now let

\[ L(E) := \bigcup_{p \in M} GL(V, E_p) \] (disjoint union).

It will shortly be clear that \( L(E) \) is a smooth manifold and the total space of a fiber bundle. Let \( \varphi : L(E) \to M \) be the projection map defined by \( \varphi(u) = p \) when \( u \in GL(V, E_p) \). Now observe that \( GL(V) \) acts on the right of the set \( L(E) \). Indeed, the action \( L(E) \times GL(V) \to L(E) \) is given by \( r : (u, g) \mapsto ug = u \circ g \).

If we pick a fixed basis for \( V \) as above then we may view \( g \) as a matrix and an element \( u \in GL(V, E_p) \) as a frame \( u = (u_1, \ldots, u_k) \). In this case we have

\[ ug = (u_1g_1^1, \ldots, u_kg_k^k) \]

It is easy to see that the orbit of a frame at \( p \) is exactly the set \( \varphi^{-1}(p) = GL(V, E_p) \) and that the action is free. For each VB-chart \( (U, \phi) \) for \( E \) let \( \sigma_\phi \) be the associated frame field. Define \( f_\phi : U \times GL(V) \to \varphi^{-1}(U) \) by \( f_\phi(p, g) = \sigma_\phi(p)g \). It is easy to check that this is a bijection. Let \( \tilde{\phi} : \varphi^{-1}(U) \to U \times GL(V) \) be the inverse of this map. We have \( \tilde{\phi} = (\varphi, \tilde{\Phi}) \) where is \( \tilde{\Phi} \) a uniquely determined map. Starting with a VB-atlas \( \{(U_\alpha, \phi_\alpha)\} \) for \( E \), we construct a family \( \{\tilde{\phi}_\alpha : \varphi^{-1}(U_\alpha) \to U_\alpha \times GL(V)\} \) of trivializations which gives a fiber bundle atlas \( \{(U_\alpha, \tilde{\phi}_\alpha)\} \) for \( L(E) \) and simultaneously induces the smooth structure.
Definition 5.1 Let $\pi : E \to M$ be a vector bundle with typical fiber $V$. The fiber bundle $(L(E), \varphi, M, GL(V))$ constructed above is called the linear frame bundle of $E$ and is usually denoted simply by $L(E)$. The frame bundle for the tangent bundle of a manifold $M$ is often denoted by $L(M)$ rather than by $L(TM)$.

Notice that any VB-atlas for $E$ induces an atlas on $L(E)$ according to our considerations above. We have

$$\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1}(p, g) = \tilde{\phi}_\alpha \sigma_\beta(p)g = \tilde{\phi}_\alpha (\Phi_{\beta|E_p} g)$$

Thus the transition functions of $L(E)$ are given by the standard transition functions of $E$ acting by left multiplication on $GL(V)$. In other words, the bundle $L(E)$ has a $GL(V)$-structure where the action of $GL(V)$ on the typical fiber $V$ is the standard one and the cocycle corresponding to the bundle atlas for $L(E)$ that we constructed from a VB-atlas $\{(U_\alpha, \phi_\alpha)\}$ for original vector bundle $E$ is the very cocycle $\Phi_{\alpha\beta}$ coming from this atlas on $E$.

We need to make one more observation concerning the right action of $GL(V)$ on $L(E)$. Take a VB-chart for $E$, say $(U, \varphi)$, and let us look again at the associated chart $(\varphi^{-1}(U), \tilde{\phi})$ for $L(E)$. First, consider the trivial bundle $p r_1 : U \times GL(V) \to U$ and define the obvious right action on the total space $\tilde{\phi} (\varphi^{-1}(U)) = U \times GL(V)$ by $((p, g_1), g) \mapsto (p, g_1 g) := (p, g_1) \cdot g$. Then this action is transitive on the fibers of this trivial bundle which are exactly the orbits of the action. Of course since $GL(V)$ acts on $L(E)$ and preserves fibers it also acts by restriction on $\varphi^{-1}(U)$.

Proposition 5.2 The bundle map $\tilde{\phi} : \varphi^{-1}(U) \to U \times GL(V)$ is equivariant with respect to the right actions described above.

Proof. We first look at the inverse.

$$\tilde{\phi}^{-1}(p, g_1)g = \sigma_\varphi(p)g_1g = \tilde{\phi}^{-1}(p, g_1g)$$

Now to see this from the point of view of $\tilde{\phi}$ rather than its inverse just let $u \in \varphi^{-1}(U) \subset L(E)$ and let $(p, g_1)$ be the unique pair such that $u = \tilde{\phi}^{-1}(p, g_1)$. Then

$$\tilde{\phi}(ug) = \tilde{\phi} (\tilde{\phi}^{-1}(p, g_1)g) = \tilde{\phi} \tilde{\phi}^{-1}(p, g_1g)$$

$$= (p, g_1g) = (p, g_1) \cdot g = \tilde{\phi}(u) \cdot g$$

A section of $L(E)$ over an open set $U$ in $M$ is just a frame field over $U$. A global frame field is a global section of $L(E)$ and clearly a global section exists if and only if $E$ is trivial.
The linear frame bundles associated to vector bundles are examples of principal bundles. For a frame bundle $F(E)$ the following things stand out: The typical fiber is diffeomorphic to the structure group $GL(V)$ and we constructed an atlas which showed that $L(E)$ had a $GL(V)$-structure where the action was left multiplication. Furthermore there is a right $GL(V)$ action on the total space $L(E)$ which has the fibers as orbits. The charts have the form $(\varphi^{-1}(U), \tilde{\phi})$ and derive from charts on $E$ and $\tilde{\phi}$ is equivariant in a sense that $\tilde{\phi}(ug) = \tilde{\phi}(u)g$ where if $\tilde{\phi}(u) = (p, g_1)$ then $(p, g_1)g := (p, g_1g)$ by definition. These facts motivate the concept of a principal bundle.

**Definition 5.3** Let $\varphi : P \to M$ be a smooth fiber bundle with typical fiber a Lie group $G$. The bundle $(P, \varphi, M, G)$ is called a principal $G$-bundle if there is a smooth free right action of $G$ on $P$ such

(i) The action preserves fibers; $\varphi(ug) = \varphi(u)$ for all $u \in P$ and $g \in G$.

(ii) For each $p \in M$ there exists a bundle chart $(U, \phi)$ with $p \in U$ and such that if $\phi = (\varphi, \Phi)$ then $\Phi(ug) = \Phi(u)g$

for all $u \in P$ and $g \in G$.

If the group $G$ is understood, then we may refer to $(P, \varphi, M, G)$ simply as a principal bundle.

Charts of the form described in (ii) in the definition are called principal bundle charts and an atlas consisting of principal bundle charts is called a principal bundle atlas. If $\varphi(u_1) = \varphi(u_2)$ then $\Phi(u_1) = \Phi(u_2)g$ where $g := \Phi(u_1)\Phi(u_2)^{-1}$ and so

$$\phi(u_1) = (\varphi(u_1), \Phi(u_1)) = (\varphi(u_2), \Phi(u_2)g) = (\varphi(u_2g), \Phi(u_2g)) = \phi(u_2g)$$

Since $\phi$ is bijective we see that $u_1 = u_2g$ and so we conclude that the fibers are actually the orbits of the right action.

Notice that if $(\phi_\alpha, U_\alpha)$ and $(\phi_\beta, U_\beta)$ are overlapping principal bundle charts with $\phi_\alpha = (\varphi, \Phi_\alpha)$ and $\phi_\beta = (\varphi, \Phi_\beta)$ then

$$\Phi_\alpha(ug)\Phi_\beta(ug)^{-1} = \Phi_\alpha(u)gg^{-1}\Phi_\beta(u)^{-1} = \Phi_\alpha(u)\Phi_\beta(u)^{-1}$$

so that the map $u \mapsto \Phi_\alpha(u)\Phi_\beta(u)^{-1}$ is constant on fibers. This means there is a smooth function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that

$$g_{\alpha\beta}(p) = \Phi_\alpha(u)\Phi_\beta(u)^{-1}$$

(5.1)

where $u$ is any element in the fiber at $p$. 
Lemma 5.4 Let \((\phi_\alpha, U_\alpha)\) and \((\phi_\beta, U_\beta)\) be overlapping principal bundle charts. For each \(p \in U_\alpha \cap U_\beta\)

\[
\Phi_\alpha|_{\varphi^{-1}(p)} \circ \left(\Phi_\beta|_{\varphi^{-1}(p)}\right)^{-1}(g) = g_{\alpha \beta}(p)g
\]

where the \(g_{\alpha \beta}\) are given as above.

**Proof.** Let \(\left(\Phi_\beta|_{\varphi^{-1}(p)}\right)^{-1}(g) = u\). Then \(g = \Phi_\beta(u)\) and so \(\Phi_\alpha|_{\varphi^{-1}(p)} \circ \Phi_\beta|_{\varphi^{-1}(p)}(g) = \Phi_\alpha(u)\). On the other hand, \(u \in \varphi^{-1}(p)\) and so

\[
g_{\alpha \beta}(p)g = \Phi_\alpha(u)\Phi_\beta(u)^{-1}g = \Phi_\alpha(u)\Phi_\beta(u)^{-1}\Phi_\beta(u)
\]

\[
= \Phi_\alpha(u) = \Phi_\alpha|_{\varphi^{-1}(p)} \circ \Phi_\beta|_{\varphi^{-1}(p)}(g)
\]

From this lemma we see that the structure group of a principal bundle is \(G\) acting on itself by left translation. Conversely if \((P, \varphi, M, G)\) is a fiber bundle with a \(G\)-atlas with \(G\) acting by left translation then \((P, \varphi, M, G)\) is a principal bundle. To see this we only need to exhibit the free right action. Let \(u \in P\) and choose a chart from \((\phi_\alpha, U_\alpha)\) the \(G\)-atlas. Then let \(ug := \phi_\alpha^{-1}(p, \Phi_\alpha(u)g)\) where \(p = \varphi(u)\). We need to show that this is well defined so let \((\phi_\beta, U_\beta)\) be another such bundle chart with \(p = \varphi(u) \in U_\beta \cap U_\alpha\). Then if \(u_1 := \phi_\beta^{-1}(p, \Phi_\beta(u)g)\) we have

\[
\phi_\alpha(u_1) = \phi_\alpha \phi_\beta^{-1}(p, \Phi_\beta(u)g) = (p, g_{\alpha \beta}(p)\Phi_\beta(u)g) = (p, \Phi_\alpha(u)g)
\]

so that \(u_1 = \phi_\alpha^{-1}(p, \Phi_\alpha(u)g) = ug\). It is easy to see that this action is free. Furthermore, since

\[
\phi_\alpha^{-1}(p, \Phi_\alpha(u)g)) = \phi_\alpha^{-1} \circ \phi_\alpha (ug)
\]

\[
= ug := \phi_\alpha^{-1}(p, \Phi_\alpha(u)g)
\]

we see that \(\Phi_\alpha(ug) = \Phi_\alpha(u)g\) as required by the definition of principal bundle.

Obviously the frame bundles of vector bundles are examples of principal bundles. We also have the Hopf Bundles described in the next example and the following exercise.

**Example 5.5 (Hopf Bundles)** Recall the Hopf map \(\varphi : S^{2n-1} \to \mathbb{C}P^{n-1}\). The quadruple \((S^{2n-1}, \varphi, \mathbb{C}P^{n-1}, U(1))\) is a principal fiber bundle. We have already defined the left action of \(U(1)\) on \(S^{2n-1}\) in Example 3.26. Since \(U(1)\) is abelian we may take this action to also be a right action. Recall that in this context, we have \(S^{2n-1} = \{\xi \in \mathbb{C}^n : |\xi| = 1\}\) where for \(\xi = (z^1, \ldots, z^n)\) we have \(|\xi| = \sum z^i z^i\). The right action of \(U(1) = S^1\) on \(S^{2n-1}\) is \((\xi, g) \mapsto \xi g = (z^1 g, \ldots, z^n g)\). It is clear that \(\varphi(\xi g) = \varphi(\xi)\). To finish the verification that \((S^{2n-1}, \varphi, \mathbb{C}P^{n-1}, U(1))\) is a principal bundle we exhibit appropriate principal
bundle charts. For each $k = 1, 2, \ldots, n$ we let $U_k := \{[z^1, \ldots, z^n] \in \mathbb{C}P^{n-1} : z^k \neq 0\}$ and we let $\psi_k : \varphi^{-1}(U_k) \rightarrow U_k \times U(1)$ be defined by $\psi_k := (\varphi, \Psi_k)$ where

$$\Psi_k(\xi) = \Psi_k(z^1, \ldots, z^n) := |z^k|^{-1} z^k$$

we leave it to the reader to show that $\psi_k := (\varphi, \Psi_k)$ is a diffeomorphism. Now we have for $g \in U(1)$

$$\Psi_k(\xi g) = |z^k g|^{-1} (z^k g) = |z^k|^{-1} (z^k g)
= \left(|z^k|^{-1} z^k\right) g = \Psi_k(\xi) g$$

as desired. Let us compute the transition cocycle $\{g_{ij}\}$. For $p = [\xi] \in U_i \cap U_i$ have

$$g_{ij}(p) = \Psi_i(\xi) \Psi_j(\xi)^{-1} = |z^i|^{-1} z^j (z^j)^{-1} |z^j| \in U(1)$$

Exercise 5.6 By analogy with the above example show that we have principal bundles $(S^{n-1}, \varphi, \mathbb{R}P^{n-1}, \mathbb{Z}_2)$ and $(S^{4n-1}, \varphi, \mathbb{H}P^{n-1}, Sp(1))$. Show that in the quaternionic case $g_{ij}(p) = |q^i|^{-1} q^j (q^j)^{-1} |q^j|$ for $p = [q^1, \ldots, q^n]$ and that the order matters in this case.

If $(U, \phi)$ is principal bundle chart for a principal bundle $(P, \varphi, M, G)$, then for each fixed $g \in G$, the map $\sigma_{\phi, g} : p \mapsto \phi^{-1}(p, g)$ is a smooth local section. Conversely if $\sigma : U \rightarrow P$ is a smooth local section then we let $f_{\sigma} : U \times G \rightarrow \varphi^{-1}(U)$ be defined by $f_{\sigma}(p, g) = \sigma(p)g$. This is a diffeomorphism and if we denote the inverse by $\phi : \varphi^{-1}(U) \rightarrow U \times G$ we have $\phi = (\varphi, \Phi)$ for a uniquely determined smooth map $\Phi : U \rightarrow G$. If $p = \varphi(u)$ we have

$$\phi(u g) = (p, \Phi(u g))$$
$$u g = \phi^{-1}(p, \Phi(u g))$$

while

$$\phi^{-1}(p, \Phi(u)g) = f_{\sigma}(p, \Phi(u)g)
= \sigma(p) (\Phi(u)g) = (\sigma(p) \Phi(u))g
= f_{\sigma}(p, \Phi(u)) g = \phi^{-1}(p, \Phi(u))g
= u g = \phi^{-1}(p, \Phi(u)g)$$

since $\phi^{-1}$ is a bijection we have $\Phi(u g) = \Phi(u)g$. Thus the section $\sigma$ has given rise to a principal bundle chart $(U, \phi)$.

Proposition 5.7 If $\varphi : P \rightarrow M$ is a surjective submersion and a Lie group $G$ acts freely on $P$ such that for each $p \in M$ the orbit of $p$ is exactly $\varphi^{-1}(p)$ then $(P, \varphi, M, G)$ is a principal bundle.
Proof. Let us assume (without loss) that the action is a right action since it can always be converted into such by group inversion if needed. We use Proposition ???: For each point \( p \in M \) there is a local section \( \sigma : U \to P \) on some neighborhood \( U \) containing \( p \). Consider the map \( f_\sigma : U \times G \to \varphi^{-1}(U) \) given by \( f_\sigma(p,g) = \sigma(p)g \). One can check that this map is injective and has an invertible tangent map at each point of \( U \). Thus by the inverse mapping theorem one may choose a possibly smaller open neighborhood of \( p \) on which \( f_\sigma \) is a fiber preserving diffeomorphism. Choose a family of local sections \( \{ \sigma_\alpha : U_\alpha \to P \} \) such that \( \cup \alpha U_\alpha = M \) and such that for each \( \alpha \) the map \( \phi_\alpha := f_{\sigma_\alpha}^{-1} : \varphi^{-1}(U_\alpha) \to U \times G \) is a fiber preserving diffeomorphism and hence bundle chart.

The same argument given before the statement of the proposition works and shows that the bundle charts \( (U_\alpha, \phi_\alpha) \) are actually principal bundle charts. \( \blacksquare \)

Combining this with our results on proper free actions we obtain the following

Corollary 5.8 If a Lie group \( G \) acts properly and freely on \( M \) (on the right) then \((M, \pi, M/G, G)\) is a principal bundle. In particular, if \( H \) is a closed subgroup of a Lie group \( G \), then \((G, \pi, G/H, H)\) is a principal bundle (with structure group \( H \)).

Definition 5.9 Let \((P_1, \varphi_1, M_1, G)\) and \((P_2, \varphi_2, M_2, G)\) be two principal \( G \)-bundles. A bundle morphism \( \tilde{f} : P_1 \to P_2 \) is called a principal \( G \)-bundle morphism if

\[
\tilde{f}(u \cdot g) = \tilde{f}(u) \cdot g
\]

for all \( g \in G \) and \( u \in P \).

Exercise 5.10 Show that if \((P_1, \varphi_1, M_1, G)\) and \((P_2, \varphi_2, M_2, G)\) are principal \( G \)-bundles and \( \tilde{f} : P_1 \to P_2 \) is a principal \( G \)-bundle morphism over a diffeomorphism \( f \) then \( \tilde{f} \) is a diffeomorphism.

If \( M_1 = M_2 = M \) and the induced map \( f \) is the identity \( \text{id}_M : M \to M \) then from the last exercise \( \tilde{f} \) is a diffeomorphism and hence a bundle equivalence (or bundle isomorphism over \( M \)) with the property \( \tilde{f}(u \cdot g) = \tilde{f}(u) \cdot g \) for all \( g \in G \) and \( u \in P \). In this case we call \( \tilde{f} \) a principal \( G \)-bundle equivalence and the two bundles are equivalent principal bundles. A principal \( G \)-bundle equivalence from a principal bundle to itself is called a principal bundle automorphism or also a (global) gauge transformation.

We have seen that a principal \( G \)-bundle atlas \( \{(U_\alpha, \phi_\alpha)\} \) is associated to a cocycle \( \{g_{\alpha\beta}\} \). From this cocycle and the left action of \( G \) on itself we may construct a bundle which has \( \{g_{\alpha\beta}\} \) as transition cocycle. In fact, recall that in the construction we formed the total space by putting an equivalence relation on the set \( \Sigma := \bigcup \alpha \{ \alpha \} \times U_\alpha \times G \) where \( (\alpha, p, g) \in \{ \alpha \} \times U_\alpha \times G \) is equivalent to \( (\beta, p', g') \in \{ \beta \} \times U_\beta \times G \) if and only if \( p = p' \) and \( g' = g_{\alpha\beta}(p) \cdot g \). Now if we define a right action on the total space of the constructed bundle by \( [\alpha, p, g_1] \cdot g = [\alpha, p, g_1 g] \) then this is well defined, smooth and makes the constructed bundle a principal \( G \)-bundle equivalent to the original principal \( G \)-bundle.
5.1. PRINCIPAL AND ASSOCIATED BUNDLES

Exercise 5.11 Prove the last assertion above.

Thus we see that $G$-cocycles on a smooth manifold $M$ give rise to principal $G$-bundles and conversely. If we start with two $G$-cocycles on $M$ then we make ask whether the principal $G$-bundles constructed from these cocycles are equivalent or not. First notice that the constructed bundles will have principal bundle atlases with the respective original transition cocycles. Thus we are led to the following related question: What conditions on the transition cocycles arising from principal bundle atlases on two principal $G$-bundles will ensure that the bundles are equivalent principal $G$-bundles? By restricting the trivializing maps to open sets of a common refinement we obtain new atlases and so we may as well assume from the start that the respective principal bundle atlases are defined on the same cover of $M$.

Proposition 5.12 Let $(P_1, \varphi_1, M, G)$ and $(P_2, \varphi_2, M, G)$ be principal $G$-bundles with principal bundle atlases $\{\{\varphi_\alpha, U_\alpha\}\}$ and $\{\{\varphi'_\alpha, U'_\alpha\}\}$ respectively. Then $(P_1, \varphi_1, M, G)$ is equivalent to $(P_2, \varphi_2, M, G)$ if and only if there exist a family of (smooth) maps $\tau_\alpha : U_\alpha \rightarrow G$ such that $g_{\alpha\beta}(p) = (\tau_\alpha(p))^{-1} g_{\alpha\beta}(p)\tau_\beta(p)$ for all $p \in U_\alpha \cap U_\beta$ and for all nonempty intersections $U_\alpha \cap U_\beta$. (Here $\{g_{\alpha\beta}\}$ is the cocycle associated to $\{\{\varphi_\alpha, U_\alpha\}\}$ and $\{g'_{\alpha\beta}\}$ is the cocycle associated to $\{\{\varphi'_\alpha, U'_\alpha\}\}$).

Sketch of Proof. First suppose that $P_1$ and $P_2$ are equivalent principal $G$-bundles and let $\tilde{f} : P_1 \rightarrow P_2$ be a bundle equivalence. Let $p \in U_\alpha$ and choose some $u \in \varphi_1^{-1}(p)$ so that $\tilde{f}(u) \in \varphi_2^{-1}(p)$. Write $\phi_\alpha = (\varphi_1, \Phi_\alpha)$ and $\phi'_\alpha = (\varphi_2, \Phi'_\alpha)$. One can easily show that $\Phi_\alpha(u)(\Phi'_\alpha(\tilde{f}(u)))^{-1}$ is an element of $G$ that is independent of the choice of $u \in \varphi_1^{-1}(p)$. Define $\tau_\alpha : U_\alpha \rightarrow G$ by

$$\tau_\alpha(p) := \Phi_\alpha(u)(\Phi'_\alpha(\tilde{f}(u)))^{-1}$$

where $u \in \varphi_1^{-1}(p)$. Do this for all $\alpha$. Suppose that $p \in U_\alpha \cap U_\beta$. Then we have $(\tau_\alpha(p))^{-1} = \phi'_\alpha(\tilde{f}(u))(\phi_\alpha(u))^{-1}$. Using the definitions of $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ (see equation 5.1) we have immediately

$$g_{\alpha\beta}(p) = (\tau_\alpha(p))^{-1} g_{\alpha\beta}(p)\tau_\beta(p)$$

Conversely, given the maps $\tau_\alpha : U_\alpha \rightarrow G$ satisfying $g_{\alpha\beta}(p) = (\tau_\alpha(p))^{-1} g_{\alpha\beta}(p)\tau_\beta(p)$ we define, for each $\alpha$, a map $f_\alpha : \varphi_1^{-1}(U_\alpha) \rightarrow \varphi_2^{-1}(U_\alpha)$ by

$$f_\alpha(u) := (\Phi'_\alpha)^{-1} \left( p, (\tau_\alpha(p))^{-1} \Phi_\alpha(u) \right)$$

Next check that $f_\alpha(p) = f_\beta(p)$ so that there is a well defined map $\tilde{f} : P_1 \rightarrow P_2$ such that $f_\alpha(p) = \tilde{f}(p)$ whenever $p \in U_\alpha$. Finally, check that $\tilde{f}(u \cdot g) = \tilde{f}(u) \cdot g$.

Let $\varphi : P \rightarrow M$ be a principal $G$-bundle and suppose that we are given a smooth left action $\lambda : G \times F \rightarrow F$ on some smooth manifold $F$. Define a right action of $G$ on $P \times F$ by

$$(u, y) \cdot g = (ug, g^{-1}y) = (ug, \lambda(g^{-1}, y))$$
Denote the orbit space of this action by $P \times_{P} F$ (or $P \times_{G} F$). Let $\tilde{\varphi} : P \times F \to P \times_{P} F$ denote the quotient map. One may check that there is a unique map $\pi : P \times_{P} F \to M$ such that $\pi \circ \tilde{\varphi} = \varphi$ and so we have a commutative diagram:

$$
\begin{array}{ccc}
P \times F & \xrightarrow{\pi} & P \\
\downarrow & & \downarrow \varphi \\
P \times_{P} F & \xrightarrow{\pi} & M
\end{array}
$$

Next we show that $(P \times_{P} F, \varphi, M, F)$ is a fiber bundle (actually a $(G, \lambda)$–bundle). It is said to be associated to the principal bundle $P$. Bundles constructed in this way are called associated bundles.

**Theorem 5.13** Referring to the above diagram and notations, $P \times_{P} F$ is a smooth manifold and

i) $(P \times_{P} F, \varphi, M, F)$ is a fiber bundle and for every principal bundle atlas $\{(U_{\alpha}, \phi_{\alpha})\}$ there is a corresponding bundle atlas $\{(U_{\alpha}, \tilde{\phi}_{\alpha})\}$ for $P \times_{P} F$ such that

$$
\tilde{\phi}_{\alpha} \circ \tilde{\phi}_{\beta}^{-1}(p, y) = (p, \lambda(g_{\alpha\beta}(p), y)) \text{ if } p \in U_{\alpha} \cap U_{\beta} \text{ and } y \in F
$$

ii) $(P \times F, \tilde{\varphi}, P \times_{P} F, G)$ is principal bundle with the right action given by $(u, y) \cdot g := (ug, g^{-1}y)$.

iii) $P \times F \xrightarrow{pr_{2}} P$ is a principal bundle morphism along $\pi$.

**Proof.** Let $\{(U_{\alpha}, \phi_{\alpha})\}$ be a principal bundle atlas for $\varphi : P \to M$. Note that $\tilde{\varphi}(\varphi^{-1}(U_{\alpha}) \times F) = \pi^{-1}(U_{\alpha})$. For each $\alpha$, define $\tilde{\Phi}_{\alpha} : \pi^{-1}(U_{\alpha}) \to F$ by requiring that $\tilde{\Phi}_{\alpha} \circ \tilde{\varphi}(u, y) = \Phi_{\alpha}(u) \cdot y$ for all $(u, y) \in \varphi^{-1}(U_{\alpha}) \times F$ and then let $\tilde{\phi}_{\alpha} := (\pi, \tilde{\Phi}_{\alpha})$ on $\pi^{-1}(U_{\alpha})$. The map $\tilde{\phi}_{\alpha}$ is clearly surjective onto $U_{\alpha} \times F$ since given $(p, y) \in U_{\alpha} \times F$ we may choose $u_{0} \in \varphi^{-1}(p)$ with $\Phi_{\alpha}(u_{0}) = e$ and then $\tilde{\phi}_{\alpha}(\tilde{\varphi}(u_{0}, y)) = (p, \phi_{\alpha}(u_{0}) \cdot y) = (p, y)$. We want to show next that $\tilde{\phi}_{\alpha}$ is injective. We define an inverse for $\tilde{\phi}_{\alpha}$. For every $p \in U_{\alpha}$ let $\sigma_{\alpha}(p) := \phi_{\alpha}^{-1}(p, e)$. Then we have

$$
\sigma_{\alpha}(p) \cdot \Phi_{\alpha}(u) = \phi_{\alpha}^{-1}(p, e) \cdot \Phi_{\alpha}(u) = \phi_{\alpha}^{-1}(p, \Phi_{\alpha}(u)) = u
$$

Define $\eta_{\alpha} : U_{\alpha} \times F \to \pi^{-1}(U_{\alpha})$ by $\eta_{\alpha}(p, y) := \tilde{\varphi}(\sigma_{\alpha}(p), y)$. Now

$$
\eta_{\alpha} \circ \tilde{\phi}_{\alpha}(\tilde{\varphi}(u, y)) = \eta_{\alpha}(p, \Phi_{\alpha}(u) \cdot y) = \tilde{\varphi}(\sigma_{\alpha}(p), \Phi_{\alpha}(u) \cdot y)
$$

$$
= \tilde{\varphi}(\sigma_{\alpha}(p) \cdot \Phi_{\alpha}(u), y) = \tilde{\varphi}(u, y)
$$

Thus $\eta_{\alpha}$ is a right inverse for $\tilde{\phi}_{\alpha}$, and so $\tilde{\phi}_{\alpha}$ is injective. It is easily checked that $\eta_{\alpha}$ is also a left inverse for $\tilde{\phi}_{\alpha}$. Indeed, $\tilde{\phi}_{\alpha} \circ \eta_{\alpha}(p, y) = (p, \tilde{\Phi}_{\alpha}(\sigma_{\alpha}(p)) \cdot y) = (p, y)$. 


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Next we check the overlaps.

\[
\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1}(p, y) = \tilde{\phi}_\alpha \circ \eta_\beta(p, y) = \tilde{\phi}_\alpha(\tilde{\rho}(\sigma_\beta(p), y)) \\
= (p, \Phi_\alpha(\sigma_\beta(p)) \cdot y) = (p, \Phi_\alpha(\tilde{\phi}_\beta^{-1}(p, e)) \cdot y) \\
= (p, \Phi_\alpha|_p \circ \Phi_\beta|_p^{-1}(e) \cdot y) \\
= (p, g_{\alpha\beta}(p) \cdot e \cdot y) = (p, g_{\alpha\beta}(p)y)
\]

This shows that the transitions mappings have the stated form and that the overlap maps \(\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1}\) are smooth. The family \(\{(U_\alpha, \phi_\alpha)\}\) provides both the induced smooth structure and is also a bundle atlas.

Since \(\tilde{\phi}_\alpha \circ \tilde{\rho}(u, p) = (\pi, \Phi_\alpha) \circ \tilde{\rho}(u, p) = (\rho(u), \Phi_\alpha(u)y)\) in the domain of every bundle chart \((U_\alpha, \phi_\alpha)\), it follows that \(\tilde{\rho}\) is smooth.

We leave it to the reader to verify that \((P \times F, \tilde{\rho}, P \times \lambda F, G)\) is a principal G-bundle. Notice that while the map \(pr_1: P \times F \rightarrow P\) is clearly a bundle map along \(\pi\) we also have \(pr_1((u, y) \cdot g) = pr_1(u \cdot g, g^{-1}y) = u \cdot g = pr_1(u, y) \cdot g\) and so that \(pr_1\) is in fact a principal bundle morphism.

We have now seen that given a principal \(G\)-bundle one may construct various fiber bundles with \(G\)-structures. Let us look at the converse situation. Suppose that \((E, \pi, M, F)\) is a fiber bundle. Suppose that this bundle has a \((G, \lambda)\)-atlas \(\{(U_\alpha, \phi_\alpha)\}\) associated with \(G\)-valued cocycle of transition functions \(\{g_{\alpha\beta}\}\). Using Theorem ?? one may construct a bundle with typical fiber \(G\) by using left translation as the action. The resulting bundle is then a principal bundle \((P, \rho, M, G)\) and it turns out that \(P \times F\) is equivalent to the original bundle \(E\). If \((E, \pi, M, V)\) a vector bundle and we use the standard \(Gl(V)\)-cocycle \(\{\Phi_{\alpha\beta}\}\) associated to a VB-atlas then the principal bundle obtained by the above construction is (equivalent to) the linear frame bundle \(F(E)\). Letting \(Gl(V)\) act on \(V\) according to the standard action we have \(F(E) \times Gl(V)V\) which is equivalent to the original bundle \((E, \pi, M, V)\). More generally, if \(\rho: G \rightarrow Gl(V)\) is a Lie group homomorphism treating \(\rho\) as a linear action we can form \(P \times \rho F\). Clearly what we have is another way of looking at bundle construction. The principal bundle takes the place of the cocycle of transition maps. For example, if we let \(Gl(V)\) act on \(V^*\) according to \(\rho(g, v) = g \cdot v := (g^{-1})^t v\) then \(F(E) \times \rho V^*\) is (equivalent to) the dual bundle \(E^*\).

5.2 Degrees of locality

There is an interplay in geometry and topology between local and global data. We now look at one aspect of this. To set up our discussion, suppose that \(s\) is a section of a smooth vector bundle \(\pi: E \rightarrow M\) with typical fiber \(V\). For simplicity we choose a basis and identify the typical fiber with \(\mathbb{R}^k\). Let us focus our attention near a point \(p\) which is contained in an open set \(U\) over which the bundle is trivialized. The trivialization provides a local frame field \(\{\sigma_1, ..., \sigma_k\}\) on \(U\). A section \(\sigma\) has a local expression \(s = \sum s^i \sigma_i\) for some smooth functions \(s^i\). Now the component functions \(s^i\) together give a smooth map \((s_i): U \rightarrow \mathbb{R}^k\).
CHAPTER 5. FIBER BUNDLES II

We may assume without loss of generality that \( U \) is the domain of a chart \( x \) for the manifold \( M \). The map \( (s_i) \circ x^{-1} : x(U) \to \mathbb{R}^k \) has a Taylor expansion centered at \( x(p) \). It will be harmless to refer to this as the Taylor expansion around \( p \). Now, we say that two sections \( \sigma_1 \) and \( \sigma_2 \) have the “same \( k \)-jet at \( p \)” if in any chart these two sections have Taylor expansions which agree up to and including terms of order \( k \). This puts an equivalence relation on sections defined near \( p \).

Consider two vector bundles \( E_1 \to M \) and \( E_2 \to M \). Suppose we have a map \( \mu : \Gamma(M, E_1) \to \Gamma(M, E_2) \) that is not necessarily linear over \( C^\infty(M) \) or even \( \mathbb{R} \). We say that \( \mu \) is \textbf{local} if the support of \( \mu(s) \) is contained in the support of \( s \) for all \( s \in \Gamma(M, E_1) \). We can ask for more refined kinds of locality. For any \( s \in \Gamma(M, E_1) \) we have a section \( \mu(s) \) and its value \( \mu(s)(p) \in E_2 \) at some point \( p \in M \). What determines the value \( \mu(s)(p) \)? Let us consider in turn the following situations:

1. It just might be the case that whenever two sections \( s_1 \) and \( s_2 \) agree on some neighborhood of any given \( p \in M \) then \( \mu(s_1)(p) = \mu(s_2)(p) \). So all that matters for determining \( \mu(s)(p) \) is the behavior of \( s \) in any arbitrarily small open set containing \( p \). To describe this we say that \( \mu(s)(p) \) only depends on the germ of \( s \) at \( p \).

2. Certainly if \( s_1 \) and \( s_2 \) agree on some neighborhood of \( p \) then they both have the same Taylor expansion at \( p \) (as seen in any local VB-charts). The reverse is not true however. Suppose that whenever two section \( s_1 \) and \( s_2 \) have Taylor series that agree up to and including terms of order \( k \) then \( \mu(s_1)(p) = \mu(s_2)(p) \). Then we say that \( \mu(s)(p) \) depends only on the \( k \)-jet of \( s \) at \( p \).

3. Finally, it might be the case that \( \mu(s_1)(p) = \mu(s_2)(p) \) exactly when \( s_1(p) = s_2(p) \).

Of course it is also possible that none of the above hold at any point. Notice that as we go down the list we are saying that the information needed to determine \( \mu(s)(p) \) is becoming more and more localized in some sense. A vector field can be viewed as an \( \mathbb{R} \)-linear map \( C^\infty(M) \to C^\infty(M) \) and since \( Xf(p) = Xg(p) \) exactly when \( df(p) = dg(p) \) we see that \( Xf(p) \) depends only on the 1-jet of \( f \) at \( p \). But this cannot be the whole story since two functions might have the same differential without sharing the same 1-jet at \( p \) since they might not agree at the 0-th jet level (if may be that \( f(p) \neq g(p) \)).

We now restrict our attention to (local) \( \mathbb{R} \)-linear maps \( L : \Gamma(M, E_1) \to \Gamma(M, E_2) \). We start at the bottom, so to speak, with 0-th order operators. One way that 0-th order operators arise is from bundle maps. If \( \tau : E_1 \to E_2 \) is a bundle map (over the identity \( M \to M \)) then we get an induced map \( \Gamma \tau : \Gamma(M, E_1) \to \Gamma(M, E_2) \) on the level of sections: If \( s \in \Gamma(M, E_1) \) then

\[
\Gamma \tau(s) := s \circ \tau.
\]
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Notice the important property that $\Gamma^\tau(fs) = f \Gamma^\tau(s)$ and so $\Gamma^\tau$ is $C^\infty(M)$ linear (a module homomorphism). Conversely, if $L: \Gamma(M, E_1) \to \Gamma(M, E_2)$ is $C^\infty(M)$ linear then $(Ls)(p)$ depends only on the value of $s(p)$ and as we see presently this means that $L$ determines a bundle $\tau$ map such that $\Gamma^\tau = L$. We shall prove a bit more general result which extends to multilinear maps.

**Proposition 5.14** Let $p \in M$ and $\tau: \Gamma(M, E_1) \times \cdots \times \Gamma(M, E_N) \to \Gamma(M, E)$ be a $C^\infty(M)$–multilinear map. Let $s_i, \bar{s}_i \in \Gamma(M, E_i)$ smooth sections such that $s_i(p) = \bar{s}_i(p)$ for $1 \leq i \leq N$; Then we have that

$$
\tau(s_1, \ldots, s_N)(p) = \tau(\bar{s}_1, \ldots, \bar{s}_N)(p)
$$

**Proof.** The proof will follow easily if we can show that $\tau(s_1, \ldots, s_N)(p) = 0$ whenever one of $s_i(p)$ is zero. We shall assume for simplicity of notation that $N = 3$. Now suppose that $s_1(p) = 0$. If $e_1, \ldots, e_N$ is a frame field over $U \subset M$ (with $p \in U$) then $s_1|_U = \sum s^i e_i$ for some smooth functions $s^i \in C^\infty(U)$. Let $\beta$ be a bump function with support in $U$. Then $\beta s_1|_U$ and $\beta^2 s_1|_U$ extend by zero to elements of $\Gamma(M, E_1)$which we shall denote by $\beta s_1$ and $\beta^2 s_1$. Similarly, $\beta e_i$ and $\beta^2 e_i$ are globally defined sections, $\beta s^i$ is a global function and $\beta s_1 = \sum \beta s^i \beta e_i$. Thus

$$
\beta^2 \tau(s_1, s_2, s_3) = \tau(\beta^2 s_1, s_2, s_3)
$$

$$
= \tau(\sum \beta s^i \beta e_i, s_2, s_3)
$$

$$
= \sum \beta \tau(s^i \beta e_i, s_2, s_3).
$$

Now since $s_1(p) = 0$ we must have $s^i(p) = 0$. Also recall that $\beta(p) = 1$. Plugging $p$ into the formula above we obtain $\tau(s_1, s_2, s_3)(p) = 0$. A similar argument holds when $s_2(p) = 0$ or $s_3(p) = 0$.

Assume that $s_i(p) = \bar{s}_i(p)$ for $1 \leq i \leq 3$. Then we have

$$
\tau(\bar{s}_1, \bar{s}_2, \bar{s}_3) - \tau(s_1, s_2, s_3)
$$

$$
= \tau(\bar{s}_1 - s_1, \bar{s}_2, \bar{s}_3) + \tau(s_1, s_2, s_3) + \tau(s_1, \bar{s}_2 - s_2, s_3)
$$

$$
+ \tau(s_1, s_2, \bar{s}_3) + \tau(\bar{s}_1, s_2, \bar{s}_3 - s_3)
$$

Since $\bar{s}_1 - s_1, \bar{s}_2 - s_2$ and $\bar{s}_3 - s_3$ are all zero at $p$ we obtain the result. ■

By the above, linearity over $C^\infty(M)$ on the level of sections corresponds to bundle maps on the vector bundle level. Thus whenever we have a $C^\infty(M)$–multilinear map $\tau: \Gamma(M, E_1) \times \cdots \times \Gamma(M, E_N) \to \Gamma(M, E)$ we also have an $\mathbb{R}$–multilinear map $E_{1p} \times \cdots \times E_{Np} \to E_p$ (which we shall often denote by the symbol $\tau_p$):

$$
\tau_p(v_1, \ldots, v_N) := \tau(s_1, \ldots, s_N)(p)
$$

for any sections $s_i$ with $s_i(p) = v_i$

The individual maps $E_{1p} \times \cdots \times E_{Np} \to E_p$ combine to give a vector bundle morphism

$$
E_1 \oplus \cdots \oplus E_N \to E
$$
If an \(\mathbb{R}\)-multilinear map \(\tau : \Gamma(M, E_1) \times \cdots \times \Gamma(M, E_N) \to \Gamma(M, E)\) is actually \(C^\infty(M)\)-linear in one or more variable then we say that \(\tau\) is \textbf{tensorial} in those variables. If \(\tau\) is tensorial in all variables say that \(\tau\) is \textbf{tensorial}.

It is worth our while to look more closely at the case \(N = 1\) above. Suppose that \(\tau : \Gamma(M, E) \to \Gamma(M, F)\) is tensorial (\(C^\infty\)-linear). Then for each \(p\) we get a linear map \(\tau_p : E_p \to F_p\) and so a bundle morphism \(E \to F\) but also we may consider the assignment \(p \mapsto \tau_p\) as a section of the bundle whose fiber at \(p\) is the space of linear transformations \(L(E_p, F_p)\) (sometimes denoted \(\text{Hom}(E_p, F_p)\)). This bundle is denoted \(L(E, F)\) or \(\text{Hom}(E, F)\) and is isomorphic to the bundle \(F \otimes E^*\).
Chapter 6

Connections and Gauge Theory
Chapter 7

Geometric Analysis

7.1 Basics

Now $E$ is a Hermitian or a Riemannian vector bundle. First, if $E$ is a Riemannian vector bundle then so is $E \otimes T^* \otimes k$ since $T^* \otimes k$ also has a metric (locally given by $g^{ij}$). If $\xi = \xi_i e^i \otimes \theta^{i_1} \otimes \cdots \otimes \theta^{i_k}$ and $\mu = \mu_i e^i \otimes \theta^{i_1} \otimes \cdots \otimes \theta^{i_k}$ then

$$\langle \xi, \mu \rangle = g^{i_1 j_1} \cdots g^{i_k j_k} \xi_i h^s \mu_s \xi^{j_1} \cdots \xi^{j_k}$$

We could denote the metric on $E \otimes T^* \otimes k$ by $h \otimes (g^{-1})^k$ but we shall not often have occasion to use this notation will stick with the notation $\langle \cdot, \cdot \rangle$ whenever it is clear which metric is meant.

Let $M, g$ be an oriented Riemannian manifold. The volume form $\text{vol}_g$ allows us to integrate smooth functions of compact support. This gives a functional $\mathcal{C}_c(M) \rightarrow \mathbb{C}$ which extends to a positive linear functional on $\mathcal{C}^0(M)$. A standard argument from real analysis applies here and gives a measure $\mu_g$ on the Borel sigma algebra $\mathcal{B}(M)$ generated by open set on $M$ which is characterized by the fact that for every open set $\mu_g(O) = \sup \left\{ \int f \, \text{vol}_g : f \prec O \right\}$ where $f \prec O$ means that $\text{supp} f$ is a compact subset of $O$ and $0 \leq f \leq 1$. We will denote integration with respect to this measure by $\int_M f(x) \mu_g(dx)$ or by a slight abuse of notation $\int_M f \, \text{vol}_g$. In fact, if $f$ is smooth then $\int_M f(x) \mu_g(dx) = \int_M f \, \text{vol}_g$. We would like to show how many of the spaces and result from analysis on $\mathbb{R}^n$ still make sense in a more global geometric setting.

7.1.1 $L^2, L^p, L^\infty$

Let $\pi : E \rightarrow M$ be a Riemannian vector bundle or a Hermitian vector bundle. Thus there is a real or complex inner product $h_p() = \langle \cdot, \cdot \rangle_p$ on every fiber $E_p$ which varies smoothly with $p \in M$. Now if $v_x \in E_x$ we let $|v_x| := \langle v_x, v_x \rangle^{1/2} \in \mathbb{R}$ and for each section $\sigma \in \Gamma(M)$ let $|\sigma| = \langle \sigma, \sigma \rangle^{1/2} \in C^0(M)$. We need a measure on the base space $M$ and so for convenience we assume that $M$ is oriented and has a Riemannian metric. Then the associated volume element $\text{vol}_g$ induces a
Radon measure which is equivalent to Lebesgue measure in every coordinate chart. We now define the norms
\[
\|\sigma\|_p := \left( \int_M |\sigma|^p \, \text{vol}_g \right)^{1/p}
\]
\[
\|\sigma\|_\infty := \sup_{x \in M} |\sigma(x)|
\]

First of all we must allow sections of the vector bundle which are not necessarily $C^\infty$. It is quite easy to see what it means for a section to be continuous and a little more difficult but still rather easy to see what it means for a section to be measurable.

**Definition 7.1** $L^p_g(M, E)$ is the space of measurable sections of $\pi : E \to M$ such that \( (\int_M |\sigma|^p \, \text{vol}_g)^{1/p} < \infty \).

With the norm $\|\sigma\|_p$ this space is a Banach space. For the case $p = 2$ we have the obvious Hilbert space inner product on $L^2_g(M, E)$ defined by
\[
(\sigma, \eta) := \int_M \langle \sigma, \eta \rangle \, \text{vol}_g
\]

Many, in fact, most of the standard facts about $L^p$ spaces of functions on a measure space still hold in this context. For example, if $\sigma \in L^p(M, E), \eta \in L^q(M, E), \frac{1}{p} + \frac{1}{q} = 1, p, q \geq 1$ then $|\sigma| \, |\eta| \in L^1(M)$ and Hölder’s inequality holds:
\[
\int_M |\sigma| \, |\eta| \, \text{vol}_g \leq \left( \int_M |\sigma|^p \, \text{vol}_g \right)^{1/p} \left( \int_M |\eta|^q \, \text{vol}_g \right)^{1/q}
\]

To what extent do the spaces $L^p_g(M, E)$ depend on the metric? If $M$ is compact, it is easy to see that for any two metrics $g_1$ and $g_2$ there is a constant $C > 0$ such that
\[
\frac{1}{C} \left( \int_M |\sigma|^p \, \text{vol}_{g_2} \right)^{1/p} \leq \left( \int_M |\sigma|^p \, \text{vol}_{g_1} \right)^{1/p} \leq C \left( \int_M |\sigma|^p \, \text{vol}_{g_2} \right)^{1/p}
\]
uniformly for all $\sigma \in L^p_g(M, E)$. Thus $L^p_g(M, E) = L^p_{g_2}(M, E)$ and the norms are equivalent. For this reason we shall forego the subscript which references the metric.

Now we add one more piece of structure into the mix.

### 7.1.2 Distributions

For every integer $m \geq 0$ and compact subset $K \subset M$ we define (semi) norm $p_{K,m}$ on the space $X^k_f(M)$ by
\[
p_{K,m}(\tau) := \sum_{j \leq m} \sup_{x \in K} \{ |\nabla^j \tau| (x) \}
\]
Let \( \mathcal{X}_K^K(K) \) denote the set of restrictions of elements of \( \mathcal{X}_K^K(M) \) to \( K \). The set \( \mathcal{X}_K^K(K) \) is clearly a vector space and \( \{p_{K,m}\}_{1 \leq m < \infty} \) is a family of norms that turns \( \mathcal{X}_K^K(K) \) into a Frechet space. Now let \( \mathcal{D}(M, \mathcal{X}_K^K) \) denote the space of \((k, l)\)-tensor fields with compact support \( \mathcal{X}_K^K(M) \subset \mathcal{X}_K^K(M) \) but equipped with the inductive limit topology of the family of Frechet spaces \( \{\mathcal{X}_K^K(K) : K \subset M \text{ compact}\} \). What we need to know is what this means in practical terms. Namely, we need a criterion for the convergence of a sequence (or net) of elements from \( \mathcal{X}_K^K(M) \).

**Criterion 7.2** Let \( \{\tau_\alpha\} \subset \mathcal{D}(M, \mathcal{X}_K^K) = \mathcal{X}_K^K(M)_c \). Then \( \tau_\alpha \to \tau \) if and only if given any \( \epsilon > 0 \) there is a compact set \( K_\epsilon \) and \( N > 0 \) such that \( \text{supp} \tau_\alpha \subset K_\epsilon \) and such that \( p_{K,m}(\tau_\alpha) < \epsilon \) whenever \( \alpha > N \).

Now the space of generalized tensors or tensor distributions \( \mathcal{D}'(M, \mathcal{X}_K^K) \) of type \((k, l)\) is the set of all linear functionals on the space \( \mathcal{D}(M, \mathcal{X}_K^K) \) which are continuous in the following sense:

**Criterion 7.3** A linear functional \( F : \mathcal{D}(M, \mathcal{X}_K^K) \to \mathbb{C} \) is continuous if and only if for every compact set \( K \subset M \) there are constants \( C(K) \) and \( m(K) \) such that

\[
|\langle F, \tau \rangle| \leq C(K)p_{K,m}(\tau) \quad \text{for all } \tau \in \mathcal{X}_K^K(K)
\]

**Definition 7.4** For each integer \( p \geq 1 \) let \( \|\tau\|_p := \left( \int_M |\tau|^p dV \right)^{1/p} \). Let \( L^p(\mathcal{X}_K^K) \) denote the completion of \( \mathcal{X}_K^K(M) \) with respect to this norm. Similarly, for every compact subset \( K \subset M \) define \( \|\tau\|_{p,K} := \left( \int_K |\tau|^p dV \right)^{1/p} \).

The family of norms \( \{\|\cdot\|_{p,K}\} \), where \( K \) runs over all compact \( K \), provides \( \mathcal{X}_K^K(M) \) with a Frechet space structure which we denote by \( \mathcal{X}_K^K(M) \).

**Definition 7.5** For each integer \( p \geq 1 \) and integer \( r \geq 0 \) let \( \|T\|_{m,p} := \sum_{|r| \leq m} \|\nabla^r T\|_p \). This defines a norm on \( \mathcal{X}_K^K(M) \) called the Sobolev norm. Let \( W_{r,p}(\mathcal{X}_K^K) \) denote the completion of \( \mathcal{X}_K^K(M) \) with respect to this norm. For \( k, l = 0, 0 \) so that we are dealing with functions we write \( W_{r,p}(M) \) instead of \( W_{r,p}(\mathcal{X}_K^K(M)) \).

### 7.1.3 Elliptic Regularity

#### 7.1.4 Star Operator II

The definitions and basic algebraic results concerning the star operator on a scalar product space globalize to the tangent bundle of a Riemannian manifold in a straightforward way.

**Definition 7.6** Let \( M, g \) be a semi-Riemannian manifold. Each tangent space is a scalar product space and so on each tangent space \( T_p M \) we have a metric volume element \( \text{vol}_p \) and then the map \( p \mapsto \text{vol}_p \) gives a section of \( \bigwedge T^* M \) called the metric volume element of \( M, g \). Also on each fiber \( \bigwedge T^*_p M \) of \( \bigwedge T^* M \)
we have a star operator \( *_p : \bigwedge^k T^*_p M \to \bigwedge^{n-k} T^*_p M \). These induce a bundle map \( * : \bigwedge^k T^* M \to \bigwedge^{n-k} T^* M \) and thus a map on sections (i.e. smooth forms) \( * : \Omega^k(M) \to \Omega^{n-k}(M) \).

**Definition 7.7** The star operator is sometimes referred to as the **Hodge star operator**.

**Definition 7.8** Globalizing the scalar product on the Grassmann algebra we get a scalar product bundle \( \Omega(M), \langle , \rangle \) where for every \( \eta, \omega \in \Omega^k(M) \) we have a smooth function \( \langle \eta, \omega \rangle \) defined by

\[
p \mapsto \langle \eta(p), \omega(p) \rangle
\]

and thus a \( C^\infty(M) \)-bilinear map \( \langle , \rangle : \Omega^k(M) \times \Omega^k(M) \to C^\infty(M) \). Declaring forms of differing degree to be orthogonal as before we extend to a \( C^\infty \) bilinear map \( \langle , \rangle : \Omega(M) \times \Omega(M) \to C^\infty(M) \).

**Theorem 7.9** For any forms \( \eta, \omega \in \Omega(M) \) we have \( \langle \eta, \omega \rangle \text{vol} = \eta \wedge * \omega \)

Now let \( M, g \) be a Riemannian manifold so that \( g = \langle , \rangle \) is positive definite. We can then define a Hilbert space of square integrable differential forms:

**Definition 7.10** Let an inner product be defined on \( \Omega_c(M) \), the elements of \( \Omega(M) \) with compact support, by

\[
(\eta, \omega) := \int_M \eta \wedge * \omega = \int_M \langle \eta, \omega \rangle \text{vol}
\]

and let \( L^2(\Omega(M)) \) denote the \( L^2 \) completion of \( \Omega_c(M) \) with respect to this inner product.

### 7.2 The Laplace Operator

The exterior derivative operator \( d : \Omega^k(M) \to \Omega^{k+1}(M) \) has a formal adjoint \( \delta : \Omega^{k+1}(M) \to \Omega^k(M) \) defined by the requirement that for all \( \alpha, \beta \in \Omega^k_c(M) \) with compact support we have

\[
(\delta \alpha, \beta) = (\alpha, \delta \beta).
\]

On a Riemannian manifold \( M \) the Laplacian of a function \( f \in C(M) \) is given in coordinates by

\[
\Delta f = -\frac{1}{\sqrt{g}} \sum_{j,k} \partial_j (g^{jk} \sqrt{g} \partial_k f)
\]

where \( g^{ij} \) is the inverse of \( g_{ij} \) the metric tensor and \( g \) is the determinant of the matrix \( G = (g_{ij}) \). We can obtain a coordinate free definition as follows. First
we recall that the divergence of a vector field $X \in \mathfrak{X}(M)$ is given at $p \in M$ by the trace of the map $\nabla X|_{T_pM}$. Here $\nabla X|_{T_pM}$ is the map

$$v \mapsto \nabla_v X.$$ 

Thus

$$\text{div}(X)(p) := \text{tr}(\nabla X|_{T_pM}).$$

Then we have

$$\Delta f := \text{div}(\text{grad}(f))$$

**Eigenvalue problem:** For a given compact Riemannian manifold $M$ one is interested in finding all $\lambda \in \mathbb{R}$ such that there exists a function $f \neq 0$ in specified subspace $S \subset L^2(M)$ satisfying $\Delta f = \lambda f$ together with certain boundary conditions in case $\partial M \neq \emptyset$.

The reader may be a little annoyed that we have not specified $S$ more clearly. The reason for this is twofold. First, the theory will work even for relatively compact open submanifolds with rather unruly topological boundary and so regularity at the boundary becomes an issue. In general, our choice of $S$ will be influenced by boundary conditions. Second, even though it may appear that $S$ must consist of $C^2$ functions, we may also seek “weak solutions” by extending $\Delta$ in some way. In fact, $\Delta$ is essentially self adjoint in the sense that it has a unique extension to a self adjoint unbounded operator in $L^2(M)$ and so eigenvalue problems could be formulated in this functional analytic setting. It turns out that under very general conditions on the form of the boundary conditions, the solutions in this more general setting turn out to be smooth functions. This is the result of the general theory of elliptic regularity.

**Definition 7.11** A boundary operator is a linear map $b : S \rightarrow C^0(\partial M)$.

Using this notion of a boundary operator we can specify boundary conditions as the requirement that the solutions lie in the kernel of the boundary map. In fact, the whole eigenvalue problem can be formulated as the search for $\lambda$ such that the linear map

$$(\Delta - \lambda) \oplus b : S \rightarrow L^2(M) \oplus C^0(\partial M)$$

has a nontrivial kernel. If we find such a $\lambda$ then this kernel is denoted $E_\lambda \subset L^2(M)$ and by definition $\Delta f = \lambda f$ and $bf = 0$ for all $f \in E_\lambda$. Such a function is called an **eigenfunction** corresponding to the eigenvalue $\lambda$. We shall see below that in each case of interest (for compact $M$) the eigenspaces $E_\lambda$ will be finite dimensional and the eigenvalues form a sequence of nonnegative numbers increasing without bound. The dimension $\text{dim}(E_\lambda)$ is called the **multiplicity** of $\lambda$. We shall present the sequence of eigenvalues in two ways:

1. If we write the sequence so as to include repetitions according to multiplicity then the eigenvalues are written as $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \uparrow \infty$. Thus it is possible, for example, that we might have $\lambda_2 = \lambda_3 = \lambda_4$ if $\text{dim}(E_{\lambda_2}) = 3$. 

2. If we wish to list the eigenvalues without repetition then we use an overbar:

\[ 0 \leq \bar{\lambda}_1 < \bar{\lambda}_2 < \ldots \uparrow \infty \]

The sequence of eigenvalues is sometimes called the **spectrum** of \( M \).

To make thing more precise we divide things up into four cases:

**The closed eigenvalue problem:** In this case \( M \) is a compact Riemannian manifold without boundary the specified subspace of \( L^2(M) \) can be taken to be \( C^2(M) \). The kernel of the map \( \Delta - \lambda : C^2(M) \to C^0(M) \) is the \( \lambda \) eigenspace and denoted by \( E_\lambda \). It consists of eigenfunctions for the eigenvalue \( \lambda \).

**The Dirichlet eigenvalue problem:** In this case \( M \) is a compact Riemannian manifold without nonempty boundary \( \partial M \). Let \( \bar{M} \) denote the interior of \( M \). The specified subspace of \( L^2(M) \) can be taken to be \( C^2(\bar{M}) \cap C^0(M) \) and the boundary conditions are \( f|\partial M \equiv 0 \) (**Dirichlet boundary conditions**) so the appropriate boundary operator is the restriction map \( b_D : f \mapsto f|\partial M \). The solutions are called **Dirichlet eigenfunctions** and the corresponding sequence of numbers \( \lambda \) for which a nontrivial solution exists is called the **Dirichlet spectrum** of \( M \).

**The Neumann eigenvalue problem:** In this case \( M \) is a compact Riemannian manifold without nonempty boundary \( \partial M \) but . The specified subspace of \( L^2(M) \) can be taken to be \( C^2(\bar{M}) \cap C^1(M) \). The problem is to find nontrivial solutions of \( \Delta f = \lambda f \) with \( f \in C^2(\bar{M}) \cap C^0(\partial M) \) that satisfy \( \nu f|\partial M \equiv 0 \) (**Neumann boundary conditions**). Thus the boundary map here is \( b_N : C^1(M) \to C^0(\partial M) \) given by \( f \mapsto \nu f|\partial M \) where \( \nu \) is a smooth unit normal vector field defined on \( \partial M \) and so the \( \nu f \) is the normal derivative of \( f \). The solutions are called **Neumann eigenfunctions** and the corresponding sequence of numbers \( \lambda \) for which a nontrivial solution exists is called the **Neumann spectrum** of \( M \).

Recall that the completion of \( C^k(M) \) (for any \( k \geq 0 \)) with respect to the inner product

\[ (f, g) = \int_M fg dV \]

is the Hilbert space \( L^2(M) \). The Laplace operator has a natural extension to a self adjoint operator on \( L^2(M) \) and a careful reformulation of the above eigenvalue problems in this Hilbert space setting together with the theory of elliptic regularity lead to the following

**Theorem 7.12** 1) For each of the above eigenvalue problems the set of eigenvalues (the spectrum) is a sequence of nonnegative numbers which increases without bound: \( 0 \leq \bar{\lambda}_1 < \bar{\lambda}_2 < \ldots \uparrow \infty \).

2) Each eigenfunction is a \( C^\infty \) function on \( M = \bar{M} \cup \partial M \).

3) Each eigenspace \( E_{\bar{\lambda}_i} \) (or \( E_D^{\bar{\lambda}_i} \) or \( E_N^{\bar{\lambda}_i} \)) is finite dimensional, that is, each eigenvalue has finite multiplicity.
4) If \( \varphi_{\lambda_1,1}, \ldots, \varphi_{\lambda_1,m_1} \) is an orthonormal basis for the eigenspace \( E_{\lambda_1} \) (or \( E_{\lambda_1}^D \) or \( E_{\lambda_1}^N \)) then the set \( B = \cup_k \{ \varphi_{\lambda_k,1}, \ldots, \varphi_{\lambda_k,m_k} \} \) is a complete orthonormal set for \( L^2(M) \). In particular, if we write the spectrum with repetitions by multiplicity, \( 0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \uparrow \infty \), then we can reindex this set of functions \( B \) as \( \{ \varphi_1, \varphi_2, \varphi_3, \ldots \} \) to obtain an ordered orthonormal basis for \( L^2(M) \) such that \( \varphi_i \) is an eigenfunction for the eigenvalue \( \lambda_i \).

The above can be given the following physical interpretation. If we think of \( M \) as a vibrating homogeneous membrane then the transverse motion of the membrane is described by a function \( f: M \times (0, \infty) \to \mathbb{R} \) satisfying
\[
\Delta f + \frac{\partial^2 f}{\partial t^2} = 0
\]
and if \( \partial M \neq \emptyset \) then we could require \( f|_{\partial M} \times (0, \infty) = 0 \) which means that we are holding the boundary fixed. A similar discussion for the Neumann boundary conditions is also possible and in this case the membrane is free at the boundary. If we look for the solutions of the form \( f(x,t) = \phi(x)T(t) \) then we are led to conclude that \( \phi \) must satisfy \( \Delta \phi = \lambda \phi \) for some real number \( \lambda \) with \( \phi = 0 \) on \( \partial M \). This is the Dirichlet eigenvalue problem discussed above.

**Theorem 7.13** For each of the eigenvalue problems defined above

Now explicit solutions of the above eigenvalue problems are very difficult to obtain except in the simplest of cases. It is interesting therefore, to see if one can tell something about the eigenvalues from the geometry of the manifold. For instance we may be interested in finding upper and/or lower bounds on the eigenvalues of the manifold in terms of various geometric attributes of the manifold. A famous example of this is the Faber–Krahn inequality which states that if \( \Omega \) is a regular domain in say \( \mathbb{R}^n \) and \( D \) is a ball or disk of the same volume then
\[
\lambda(\Omega) \geq \lambda(D)
\]
where \( \lambda(\Omega) \) and \( \lambda(D) \) are the lowest nonzero Dirichlet eigenvalues of \( \Omega \) and \( D \) respectively. Now it is of interest to ask whether one can obtain geometric information about the manifold given a degree of knowledge about the eigenvalues. There is a 1966 paper by M. Kac entitled “Can One Hear the Shape of a Drum?” which addresses this question. Kac points out that Weyl’s asymptotic formula shows that the sequence of eigenvalues does in fact determine the volume of the manifold. Weyl’s formula is
\[
(\lambda_k)_{n/2} \sim \left( \frac{(2\pi)^n}{\omega_n} \right) \frac{k}{\text{vol}(M)} \quad \text{as} \quad k \to \infty
\]
where \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \) and \( M \) is the given compact manifold. In particular,
\[
\left(\frac{(2\pi)^n}{\omega_n}\right) \lim_{k \to \infty} \frac{k}{(\lambda_k)^{n/2}} = \text{vol}(M).
\]

So the volume is indeed determined by the spectrum\(^1\).

### 7.3 Spectral Geometry

Legend has it that Pythagoras was near a black-smith’s shop one day and hearing the various tones of a hammer hitting an anvil was lead to ponder the connection between the geometry (and material composition) of vibrating objects and the pitches of the emitted tones. This lead him to experiment with vibrating strings and a good deal of mathematics ensued. Now given a string of uniform composition it is essentially the length of the string that determines the possible pitches. Of course, there isn’t much Riemannian geometry in a string because the dimension is 1. Now we have seen that a natural mathematical setting for vibration in higher dimensions is the Riemannian manifold and the wave equation associated with the Laplacian. The spectrum of the Laplacian corresponds the possible frequencies of vibration and it is clearly only the metric together with the total topological structure of the manifold that determines the spectrum. If the manifold is a Lie group or is a homogeneous space acted on by a Lie group, then the topic becomes highly algebraic but simultaneously involves fairly heavy analysis. This is the topic of harmonic analysis and is closely connected with the study of group representations. One the other hand, the Laplacian and its eigenvalue spectrum are defined for arbitrary (compact) Riemannian manifolds and, generically, a Riemannian manifold is far from being a Lie group or homogeneous space. The isometry group may well be trivial. Still the geometry must determine the spectrum. But what is the exact relationship between the geometry of the manifold and the spectrum? Does the spectrum determine the geometry? Is it possible that two manifolds can have the same spectrum without being isometric to each other? That the answer is yes has been known for quite a while now it wasn’t until (??whenn) that the question was answered for planar domains? This was Mark Kac’s original question: “Can one hear the shape of a drum?” It was shown by Carolyn Gordon and (??WhoW?) that the following two domains have the same Dirichlet spectrum but are not isometric:

\[
\text{finish}
\]

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\(^1\) Notice however, one may ask still how far out into the spectrum must one “listen” in order to gain an estimate of \(\text{vol}(M)\) to a given accuracy.
7.4 Hodge Theory

7.5 Dirac Operator

It is often convenient to consider the differential operator $D = i \frac{\partial}{\partial x}$ instead of $\frac{\partial}{\partial x}$ even when one is interested mainly in real valued functions. For one thing $D^2 = - \frac{\partial^2}{\partial x^2}$ and so $D$ provides a sort of square root of the positive Euclidean Laplacian $\Delta = - \frac{\partial^2}{\partial x^2}$ in dimension 1. Dirac wanted a similar square root for the wave operator $\Box = \frac{\partial^2}{\partial x^2} - \sum_{i=1}^3 \gamma_i \partial_i$ (the Laplacian in $\mathbb{R}^4$ for the Minkowski inner metric) and found that an operator of the form $D = \partial_0 - \sum_{i=1}^3 \gamma_i \partial_i$ would do the job if it could be arranged that $\gamma_i \gamma_j + \gamma_j \gamma_i = 2 \delta_{ij}$ where

$$
(n_{ij}) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{bmatrix}.
$$

One way to do this is to allow the $\gamma_i$ to be matrices.

Now let’s consider finding a square root for $\Delta = - \sum_{i=1}^n \partial^2_i$. We accomplish this by an $\mathbb{R}$-linear embedding of $\mathbb{R}^n$ into an $N \times N$ real or complex matrix algebra $A$ by using $n$ linearly independent matrices $\{\gamma_i : i = 1, 2, \ldots, n\}$ (so called “gamma matrices”) and mapping

$$(x^1, \ldots, x^n) \mapsto x^i \gamma_i \text{ (sum)}.$$ 

and where $\gamma_1, \ldots, \gamma_n$ are matrices satisfying the basic condition

$$\gamma_i \gamma_j + \gamma_j \gamma_i = -2 \delta_{ij}.$$ 

We will be able to arrange\(^2\) that $\{1, \gamma_1, \ldots, \gamma_n\}$ generates an algebra of dimension $2^n$ spanned as vector space by the identity matrix 1 and all products of the form $\gamma_{i_1} \cdots \gamma_{i_k}$ with $i_1 < i_2 < \cdots < i_k$. Thus we aim to identify $\mathbb{R}^n$ with the linear span of these gamma matrices. Now if we can find matrices with the property that $\gamma_i \gamma_j + \gamma_j \gamma_i = -2 \delta_{ij}$ then our “Dirac operator” will be

$$D = \sum_{i=1}^n \gamma_i \partial_i$$

which is now acting on $N$-tuples of smooth functions.

Now the question arises: What are the differential operators $\partial_i = \frac{\partial}{\partial x^i}$ acting on exactly. The answer is that they act on whatever we take the algebra spanned by the gamma matrices to be acting on. In other words we should have some vector space $S$ that is a module over the algebra spanned by the gamma matrices.

\(^2\)It is possible that gamma matrices might span a space of half the dimension we are interested in. This fact has gone unnoticed in some of the literature. The dimension condition is to assure that we get a universal Clifford algebra.
Then we take as our “fields” smooth maps $f : \mathbb{R}^n \to S$. Of course since the $\gamma_i \in M_{N\times N}$ we may always take $S = \mathbb{R}^N$ with the usual action of $M_{N\times N}$ on $\mathbb{R}^N$. The next example shows that there are other possibilities.

**Example 7.14** Notice that with $\frac{\partial}{\partial z} := \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial \bar{z}} := \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$ we have

$$\left[ \begin{array}{cc} 0 & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{array} \right] \left[ \begin{array}{cc} 0 & -\frac{\partial}{\partial \bar{z}} \\ \frac{\partial}{\partial \bar{z}} & 0 \end{array} \right] = \left[ \begin{array}{cc} \triangle & 0 \\ 0 & \triangle \end{array} \right]$$

where $\triangle = -\sum \partial_i^2$. On the other hand

$$\left[ \begin{array}{cc} 0 & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{array} \right] = \left[ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right] \frac{\partial}{\partial x} + \left[ \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right] \frac{\partial}{\partial y}.$$

From this we can see that appropriate gamma matrices for this case are $\gamma_1 = \left[ \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right]$ and $\gamma_2 = \left[ \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right]$.

Now let $E^0$ be the span of $1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$ and $\gamma_2 \gamma_1 = \left( \begin{array}{cc} -i & 0 \\ 0 & -i \end{array} \right)$. Let $E^1$ be the span of $\gamma_1$ and $\gamma_2$. Refer to $E^0$ and $E^1$ the even and odd parts of $\text{Span}\{1, \gamma_1, \gamma_2, \gamma_2 \gamma_1 \}$. Then we have that $D = \left[ \begin{array}{cc} 0 & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{array} \right]$ maps $E^0$ to $E^1$ and writing a typical element of $E^0$ as $f(x, y) = u(x, y) + \gamma_2 \gamma_1 v(x, y)$ is easy to show that $Df = 0$ is equivalent to the Cauchy-Riemann equations.

The reader should keep this last example in mind as this kind of decomposition into even and odd part will be a general phenomenon below.

### 7.5.1 Clifford Algebras

A Clifford algebra is the type of algebraic object that allows us to find differential operators that square to give Laplace type operators. The matrix approach described above is in fact quite general but there are other approaches that are more abstract and encourage one to think about a Clifford algebra as something that contains the scalar and the vector space we start with. The idea is similar to that of the complex numbers. We seldom think about complex numbers as “pairs of real numbers” while we are calculating unless push comes to shove. After all, there are other good ways to represent complex numbers; as matrices for example. And yet there is one underlying abstract object called the complex numbers which is quite concrete once one gets used to using them. Similarly we encourage the reader to learn to think about abstract Clifford algebras in the same way. Just compute!

Clifford algebras are usually introduced in connection with a quadratic form $q$ on some vector space but in fact we are just as interested in the associated
symmetric bilinear form and so in this section we will generically use the same symbol for a quadratic form and the bilinear form obtained by polarization and write both \( q(v) \) and \( q(v, w) \).

**Definition 7.15** Let \( V \) be an \( n \) dimensional vector space over a field \( \mathbb{K} \) with characteristic not equal to 2. Suppose that \( q \) is a quadratic form on \( V \) and let \( q^\prime \) be the associated symmetric bilinear form obtained by polarization. A **Clifford algebra based on** \( V, q \) is an algebra with unity \( 1_{\mathcal{C}l}(V, q, \mathbb{K}) \) containing \( V \) (or an isomorphic image of \( V \)) such that the following relations hold:

\[
vw + wv = -2q(v, w)1
\]

and such that \( \mathcal{C}l(V, q, \mathbb{K}) \) is universal in the following sense: Given any linear map \( L: V \to A \) into an associative \( \mathbb{K} \)-algebra with unity \( 1 \) such that

\[
L(v)L(w) + L(w)L(v) = -2q(v, w)1
\]

then there is a unique extension of \( L \) to an algebra homomorphism \( \overline{L}: \mathcal{C}l(V, q, \mathbb{K}) \to A \).

If \( e_1, \ldots, e_n \) is an orthonormal basis for \( V, q \) then we must have

\[
e_i e_j + e_j e_i = 0 \quad \text{for} \quad i \neq j
\]

\[
e_i^2 = -q(e_i) = \pm 1 \quad \text{or} \quad 0
\]

A common choice is the case when \( q \) is a nondegenerate inner product on a real vector space. In this case we have a particular realization of the Clifford algebra obtained by introducing a new product into the Grassmann vector space \( \wedge V \). The said product is the unique linear extension of the following rule for \( v \in \wedge^1 V \) and \( w \in \wedge^k V \):

\[
v \cdot w := v \wedge w - v^\flat \lrcorner w
\]

\[
w \cdot v := (-1)^k(v \wedge w + v^\flat \lrcorner w)
\]

We will refer to this as a **geometric algebra** on \( \wedge V \) and this version of the Clifford algebra will be called the **form presentation of** \( \mathcal{C}l(V, q) \). Now once we have a definite inner product on \( V \) we have an inner product on \( V^* \) and \( V \cong V^* \). The Clifford algebra on \( V^* \) is generated by the following more natural looking formulas

\[
\alpha \cdot \beta := \alpha \wedge \beta - (\sharp \alpha) \cdot \beta
\]

\[
\beta \cdot \alpha := (-1)^k(\alpha \wedge \beta + (\sharp \alpha) \cdot \beta)
\]

for \( \alpha \in \wedge^1 V \) and \( \beta \in \wedge V \).

Now we have seen that one can turn \( \wedge V \) (or \( \wedge V^* \)) into a Clifford algebra and we have also seen that one can obtain a Clifford algebra whenever appropriate gamma matrices can be found. A slightly more abstract construction is also
common: Denote by \( I(q) \) the ideal of the full tensor algebra \( T(V) \) generated by elements of the form \( x \otimes x - q(x) \cdot 1 \). The **Clifford algebra** is (up to isomorphism) given by

\[
\text{Cl}(V,q,\mathbb{K}) = T(V)/I(q).
\]

We can use the canonical injection

\[ i : V \rightarrow C_{\mathbb{K}} \]

to identify \( V \) with its image in \( \text{Cl}(V,q,\mathbb{K}) \). (The map turns out that \( i \) is 1–1 onto \( i(V) \) and we will just accept this without proof.)

**Exercise 7.16** Use the universal property of \( \text{Cl}(V,q,\mathbb{K}) \) to show that it is unique up to isomorphism.

**Remark 7.17** Because of the form realization of a Clifford algebra we see that \( \wedge V \) is a \( \text{Cl}(V,q,\mathbb{R}) \)–module. But even if we just have some abstract \( \text{Cl}(V,q,\mathbb{R}) \) we can use the universal property to extend the action of \( V \) on \( \wedge V \) given by

\[ v \mapsto v \cdot w := v \wedge w - v^\flat \lrcorner w \]

to an action of \( \text{Cl}(V,q,\mathbb{K}) \) on \( \wedge V \) thus making \( \wedge V \) a \( \text{Cl}(V,q,\mathbb{R}) \)–module.

**Definition 7.18** Let \( \mathbb{R}^n_{(r,s)} \) be the real vector space \( \mathbb{R}^n \) with the inner product of signature \( (r,s) \) given by

\[ (x,y) := \sum_{i=1}^{r} x_i y_i - \sum_{i=r+1}^{r+s} x_i y_i. \]

The Clifford algebra formed from this inner product space is denoted \( \text{Cl}_{r,s} \). In the special case of \( (p,q) = (n,0) \) we write \( \text{Cl}_n \).

**Definition 7.19** Let \( \mathbb{C}^n \) be the complex vector space of \( n \)-tuples of complex numbers together with the standard symmetric \( \mathbb{C} \)–bilinear form

\[ b(z,w) := \sum_{i=1}^{n} z_i w_i. \]

The (complex) Clifford algebra obtained is denoted \( \text{Cl}_n \).

**Remark 7.20** The complex Clifford algebra \( \text{Cl}_n \) is based on a complex symmetric form and not on a Hermitian form.

**Exercise 7.21** Show that for any nonnegative integers \( p,q \) with \( p + q = n \) we have \( \text{Cl}_{p,q} \otimes \mathbb{C} \cong \text{Cl}_n \).

**Example 7.22** The Clifford algebra based on \( \mathbb{R}^1 \) itself with the relation \( x^2 = -1 \) is just the complex number system.
7.5. **Dirac Operator**

The Clifford algebra construction can be globalized in the obvious way. In particular, we have the option of using the form presentation so that the above formulas $\alpha \cdot \beta := \alpha \wedge \beta - (\sharp \alpha) \beta$ and $\beta \cdot \alpha := (-1)^{k}(\alpha \wedge \beta + (\sharp \alpha) \beta)$ are interpreted as equations for differential forms $\alpha \in \wedge^{1}T^{*}M$ and $\beta \in \wedge^{k}T^{*}M$ on a semi-Riemannian manifold $M, g$. In any case we have the following

**Definition 7.23** Given a Riemannian manifold $M, g$, the Clifford algebra bundle is $\text{Cl}(T^{*}M, g) = \text{Cl}(T^{*}M) := \cup_{x} \text{Cl}(T^{*}_{x}M)$.

Since we take each tangent space to be embedded $T^{*}_{x}M \subset \text{Cl}(T^{*}_{x}M)$, the elements $\theta^{i}$ of a local orthonormal frame $\theta^{1}, \ldots, \theta^{n} \in \Omega^{1}$ are also local sections of $\text{Cl}(T^{*}M, g)$ and satisfy

$$\theta^{i} \theta^{j} + \theta^{j} \theta^{i} = -\langle \theta^{i}, \theta^{j} \rangle = -\delta^{ij}$$

Recall that $e^{1}, \ldots, e^{n}$ is a list of numbers equal to $\pm 1$ (or even 0 if we allow degeneracy) and giving the index of the metric $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle$.

Obviously, we could also work with the bundle $\text{Cl}(TM) := \cup_{x} \text{Cl}(T_{x}M)$ which is naturally isomorphic to $\text{Cl}(T^{*}M)$ in which case we would have

$$e^{i} e^{j} + e^{j} e^{i} = -\langle e^{i}, e^{j} \rangle = -\delta^{ij}$$

for orthonormal frames. Of course it shouldn’t make any difference to our development since one can just identify $TM$ with $T^{*}M$ by using the metric. On the other hand, we could define $\text{Cl}(T^{*}M, b)$ even if $b$ is a degenerate bilinear tensor and then we recover the Grassmann algebra bundle $\wedge T^{*}M$ in case $b \equiv 0$. These comments should make it clear that $\text{Cl}(T^{*}M, g)$ is in general a sort of deformation of the Grassmann algebra bundle.

There are a couple of things to notice about $\text{Cl}(T^{*}M)$ when we realize it as $\wedge T^{*}M$ with a new product. First of all if $\alpha, \beta \in \wedge T^{*}M$ and $\langle \alpha, \beta \rangle = 0$ then $\alpha \cdot \beta = \alpha \wedge \beta$ where as if $\langle \alpha, \beta \rangle \neq 0$ then in general $\alpha \beta$ is not a homogeneous element. Second, $\text{Cl}(T^{*}M)$ is locally generated by $\{1\} \cup \{\theta^{1}\} \cup \{\theta^{i} \theta^{j} : i < j\} \cup \cdots \cup \{\theta^{1} \theta^{2} \cdots \theta^{n}\}$ where $\theta^{1}, \theta^{2}, \ldots, \theta^{n}$ is a local orthonormal frame. Now we can immediately define our current objects of interest:

**Definition 7.24** A bundle of modules over $\text{Cl}(T^{*}M)$ is a vector bundle $\Sigma = (E, \pi, M)$ such that each fiber $E_{x}$ is a module over the algebra $\text{Cl}(T^{*}_{x}M)$ and such that for each $\theta \in \Gamma(\text{Cl}(T^{*}M))$ and each $\sigma \in \Gamma(\Sigma)$ the map $x \mapsto \theta(x) \sigma(x)$ is smooth. Thus we have an induced map on smooth sections: $\Gamma(\text{Cl}(T^{*}M)) \times \Gamma(\Sigma) \to \Gamma(\Sigma)$.

**Proposition 7.25** The bundle $\text{Cl}(T^{*}M)$ is a Clifford module over itself and the Levi-Civita connection $\nabla$ on $M$ induces a connection on $\text{Cl}(T^{*}M)$ (this connection is also denoted $\nabla$) such that

$$\nabla(\sigma_{1} \sigma_{2}) = (\nabla \sigma_{1}) \sigma_{2} + \sigma_{1} \nabla \sigma_{2}$$

for all $\sigma_{1}, \sigma_{2} \in \Gamma(\text{Cl}(T^{*}M))$. In particular, if $X, Y \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(\text{Cl}(T^{*}M))$ then

$$\nabla_{X}(Y \sigma) = (\nabla_{X} Y) \sigma + Y \nabla_{X} \sigma.$$
Let $\text{Cl}(T^*M)$ as $\wedge T^*M$ with Clifford multiplication and let $\nabla$ be usual induced connection on $\wedge T^*M \subset \otimes T^*M$. We have for an local orthonormal frame $e_1, \ldots, e_n$ with dual frame $\theta^1, \theta^2, \ldots, \theta^n$. Then $\nabla_\xi \theta^i = -\Gamma^j_i(\xi) \theta^j$

\[
\nabla_\xi (\theta^i \theta^j) = \nabla_\xi (\theta^i \wedge \theta^j) = \nabla_\xi \theta^i \wedge \theta^j + \theta^i \wedge \nabla_\xi \theta^j = -\Gamma^k_i(\xi) \theta^k \wedge \theta^j - \Gamma^k_j(\xi) \theta^k \wedge \theta^i = -\Gamma^k_i(\xi) \theta^k \theta^j - \Gamma^k_j(\xi) \theta^k \theta^i = (\nabla_\xi \theta^i) \theta^j + \theta^i \nabla_\xi \theta^j
\]

The result follows by linearity and a simple induction since a general section $\sigma$ can be written locally as $\sigma = \sum a_{i_1 \ldots i_k} \theta^{i_1} \theta^{i_2} \ldots \theta^{i_k}$. □

**Definition 7.26** Let $M, g$ be a (semi-) Riemannian manifold. A compatible connection for a bundle of modules $\Sigma$ over $\text{Cl}(T^*M)$ is a connection $\nabla^{\Sigma}$ on $\Sigma$ such that

\[
\nabla^{\Sigma}(\sigma \cdot s) = (\nabla \sigma) \cdot s + \sigma \cdot \nabla^{\Sigma} s
\]

for all $s \in \Gamma(\Sigma)$ and all $\sigma \in \Gamma(\text{Cl}(T^*M))$.

**Definition 7.27** Let $\Sigma = (E, \pi, M)$ be a bundle of modules over $\text{Cl}(T^*M)$ with a compatible connection $\nabla = \nabla^{\Sigma}$. The associated Dirac operator is defined as a differential operator $\Sigma$ on by

\[
D_s := \sum \theta^i \cdot \nabla^{\Sigma}_{e_i} s
\]

for $s \in \Gamma(\Sigma)$.

Notice that Clifford multiplication of $\text{Cl}(T^*M)$ on $\Sigma = (E, \pi, M)$ is a zeroth order operator and so is well defined as a fiberwise operation $\text{Cl}(T^*_x M) \times E_x \to E_x$.

There are still a couple of convenient properties that we would like to have. These are captured in the next definition.

**Definition 7.28** Let $\Sigma = (E, \pi, M)$ be a bundle of modules over $\text{Cl}(T^*M)$ such that $\Sigma$ carries a Riemannian metric and compatible connection $\nabla = \nabla^{\Sigma}$. We call $\Sigma = (E, \pi, M)$ a Dirac bundle if the following equivalent conditions hold:

1) $(e s_1, e s_2) = (s_1, s_2)$ for all $s_1, s_2 \in E_x$ and all $e \in T^*_x M \subset \text{Cl}(T^*_x M)$ with $|e| = 1$. In other words, Clifford multiplication by a unit (co)vector is required to be an isometry of the Riemannian metric on each fiber of $\Sigma$. Since, $e^2 = -1$ it follows that this is equivalent to requiring $-

2) (e s_1, e s_2) = -(s_1, e s_2)$ for all $s_1, s_2 \in E_x$ and all $e \in T^*_x M \subset \text{Cl}(T^*_x M)$ with $|e| = 1$.

Assume in the sequel that $q$ is nondegenerate. Let denote the subalgebra generated by all elements of the form $x_1 \cdots x_k$ with $k$ even. And similarly, $\text{Cl}_k(V, q)$, with $k$ odd. Thus $\text{Cl}(V, q)$ has the structure of a $\mathbb{Z}_2$-graded algebra:

\[
\text{Cl}(V, q) = \text{Cl}_0(V, q) \oplus \text{Cl}_1(V, q)
\]
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\[ \text{Cl}_0(V,q) \cdot \text{Cl}_0(V,q) \subset \text{Cl}_0(V,q) \]
\[ \text{Cl}_0(V,q) \cdot \text{Cl}_1(V,q) \subset \text{Cl}_1(V,q) \]
\[ \text{Cl}_1(V,q) \cdot \text{Cl}_1(V,q) \subset \text{Cl}_0(V,q) \]

\( \text{Cl}_0(V,q) \) and \( \text{Cl}_1(V,q) \) are referred to as the even and odd part respectively. A \( \mathbb{Z}_2 \)-graded algebra is also called a superalgebra. There exists a fundamental automorphism \( \alpha \) of \( \text{Cl}(V,q) \) such that \( \alpha(x) = -x \) for all \( x \in V \). Note that \( \alpha^2 = \text{id} \). It is easy to see that \( \text{Cl}_0(V,q) \) and \( \text{Cl}_1(V,q) \) are the +1 and \( -1 \) eigenspaces of \( \alpha : \text{Cl}(V,q) \to \text{Cl}(V,q) \).

7.5.2 The Clifford group and spinor group

Let \( G \) be the group of all invertible elements \( s \in C_K \) such that \( sVs^{-1} = V \). This is called the Clifford group associated to \( q \). The special Clifford group is \( G^+ = G \cap C_0 \). Now for every \( s \in G \) we have a map \( \phi_s : v \mapsto sv{s}^{-1} \) for \( v \in V \). It can be shown that \( \phi \) is a map from \( G \) into \( O(q) \), the orthogonal group of \( q \). The kernel is the invertible elements in the center of \( C_K \).

It is a useful and important fact that if \( x \in G \cap V \) then \( q(x) \neq 0 \) and \( -\phi_x \) is reflection through the hyperplane orthogonal to \( x \). Also, if \( s \) is in \( G^+ \) then \( \phi_s \) is in \( SO(q) \). In fact, \( \phi(G^+) = SO(q) \).

Besides the fundamental automorphism \( \alpha \) mentioned above, there is also a fundamental anti-automorphism or reversion \( \beta : \text{Cl}(V,q) \to \text{Cl}(V,q) \) which is determined by the requirement that \( \beta(v_1v_2 \cdots v_k) = v_kv_{k-1} \cdots v_1 \) for \( v_1, v_2, ..., v_k \in V \subseteq \text{Cl}(V,q) \). We can use this anti-automorphism \( \beta \) to put a kind of “norm” on \( G^+ \):

\[ N : G^+ \to K^* \]

where \( K^* \) is the multiplicative group of nonzero elements of \( K \) and \( N(s) = \beta(s)s \). This is a homomorphism and if we “mod out” the kernel of \( N \) we get the so-called reduced Clifford group \( G_0^+ \).

We now specialize to the real case \( K = \mathbb{R} \). The identity component of \( G_0^+ \) is called the spin group and is denoted by \( \text{Spin}(V,q) \).

7.6 The Structure of Clifford Algebras

Now if \( K = \mathbb{R} \) and

\[ q(x) = \sum_{i=1}^{r} (x_i)^2 - \sum_{i=r+1}^{r+s} (x_i)^2 \]

we write \( C(\mathbb{R}^{r+s}, q, \mathbb{R}) = \text{Cl}(r,s) \). Then one can prove the following isomorphisms.

\[ \text{Cl}(r+1, s+1) \cong \text{Cl}(1,1) \otimes C(r,s) \]
\[ \text{Cl}(s+2, r) \cong \text{Cl}(2,0) \otimes C(r,s) \]
\[ \text{Cl}(s, r+2) \cong \text{Cl}(0,2) \otimes C(r,s) \]
and

\[ Cl(p, p) \cong \bigotimes_p Cl(1, 1) \]

\[ Cl(p + k, p) \cong \bigotimes_p Cl(1, 1) \bigotimes Cl(k, 0) \]

\[ Cl(k, 0) \cong Cl(2, 0) \otimes Cl(0, 2) \otimes Cl(k - 4, 0) \quad k > 4 \]

Using the above type of periodicity relations together with

\[ Cl(2, 0) \cong Cl(1, 1) \cong M_2(\mathbb{R}) \]

\[ Cl(1, 0) \cong \mathbb{R} \oplus \mathbb{R} \]

and

\[ Cl(0, 1) \cong \mathbb{C} \]

we can piece together the structure of \( Cl(r, s) \) in terms of familiar matrix algebras. We leave out the resulting table since for one thing we are more interested in the simpler complex case. Also, we will explore a different softer approach below.

The complex case. In the complex case we have a much simpler set of relations;

\[ Cl(2r) \cong Cl(r, r) \otimes \mathbb{C} \cong M_{2^r}(\mathbb{C}) \]

\[ Cl(2r + 1) \cong Cl(1) \otimes Cl(2r) \]

\[ \cong Cl(2r) \oplus Cl(2r) \cong M_{2^r}(\mathbb{C}) \oplus M_{2^r}(\mathbb{C}) \]

These relations remind us that we may use matrices to represent our Clifford algebras. Let’s return to this approach and explore a bit.

### 7.6.1 Gamma Matrices

**Definition 7.29** A set of real or complex matrices \( \gamma_1, \gamma_2, ..., \gamma_n \) are called gamma matrices for \( Cl(r, s) \) if

\[ \gamma_i \gamma_j + \gamma_j \gamma_i = -2g_{ij} \]

where \( (g_{ij}) = \text{diag}(1, ..., 1, -1, ..., ) \) is the diagonalized matrix of signature \( (r, s) \).

**Example 7.30** Recall the Pauli matrices:

\[ \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

It is easy to see that \( \sigma_1, \sigma_2 \) serve as gamma matrices for \( Cl(0, 2) \) while \( -i\sigma_1, -i\sigma_2 \) serve as gamma matrices for \( Cl(2, 0) \).
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Cl(2,0) is spanned as a vector space of matrices by $\sigma_0, -i\sigma_1, -i\sigma_2, -i\sigma_3$ and is (algebra) isomorphic to the quaternion algebra $\mathbb{H}$ under the identification

$\sigma_0 \mapsto 1$
$-i\sigma_1 \mapsto i$
$-i\sigma_2 \mapsto J$
$-i\sigma_3 \mapsto K$

7.7 Clifford Algebra Structure and Representation

7.7.1 Bilinear Forms

We will need some basic facts about bilinear forms. We review this here.

Let $E$ be a module over a commutative ring $R$. Typically $E$ is a vector space over a field $K$. A bilinear map $g : E \times E \rightarrow R$ is called symmetric if $g(x, y) = g(y, x)$ and antisymmetric if $g(x, y) = -g(y, x)$ for $(x, y) \in E \times E$. If $R$ has an automorphism of order two, $a \mapsto \bar{a}$ we say that $g$ is Hermitian if $g(ax, y) = \bar{a}g(x, y)$ and $g(x, ay) = \bar{a}g(x, y)$ for all $a \in R$ and $(x, y) \in E \times E$. If $g$ is any of symmetric, antisymmetric, or Hermitian then the “left kernel” of $g$ is equal to the “right kernel”. That is

$$\ker g = \{ x \in E : g(x, y) = 0 \quad \forall y \in E \}$$
$$= \{ y \in E : g(x, y) = 0 \quad \forall x \in E \}$$

If $\ker g = 0$ we say that $g$ is nondegenerate. In case $E$ is a vector space of finite dimension $g$ is nondegenerate if and only if $x \mapsto g(x, \cdot) \in E^*$ is an isomorphism. An orthogonal basis for $g$ is a basis $\{v_i\}$ for $E$ such that $g(v_i, v_j) = 0$ for $i \neq j$.

**Definition 7.31** Let $E$ be a vector space over a three types above. If $E = E_1 \oplus E_2$ for subspaces $E_i \subset E$ and $g(x_1, x_2) = 0 \quad \forall x_1 \in E, x_2 \in E_2$ then we write

$$E = E_1 \perp E_2$$

and say that $E$ is the orthogonal direct sum of $E_1$ and $E_2$.

**Proposition 7.32** Suppose $E, g$ is as above with

$$E = E_1 \perp E_2 \perp \cdots \perp E_k$$

Then $g$ is non-degenerate if and only if its restrictions $g|_{E_i}$ are and

$$\ker g = E_1^0 \perp E_2^0 \perp \cdots \perp E_k^0$$

**Proof.** Nearly obvious. ■

**Terminology:** If $g$ is one of symmetric, antisymmetric or Hermitian we say that $g$ is geometric.
Proposition 7.33  Let \( g \) be a geometric bilinear form on a vector space \( E \) (over \( K \)). Suppose \( g \) is nondegenerate. Then \( g \) is nondegenerate on a subspace \( F \) if and only if \( E = F \perp F^{\perp} \) where

\[
F^{\perp} = \{ x \in E : g(x, f) = 0 \quad \forall f \in F \}
\]

Definition 7.34  A map \( q \) is called quadratic if there is a symmetric \( g \) such that \( q(x) = g(x, x) \). Note that \( g \) can be recovered from \( q \):

\[
2g(x, y) = q(x + y) - q(x) - q(y)
\]

7.7.2 Hyperbolic Spaces And Witt Decomposition

\( E, g \) is a vector space with symmetric form \( g \). If \( E \) has dimension 2 we call \( E \) a hyperbolic plane. If \( \dim E \geq 2 \) and \( E = E_1 \perp E_2 \perp \cdots \perp E_k \) where each \( E_i \) is a hyperbolic plane for \( g|_{E_i} \) then we call \( E \) a hyperbolic space. For a hyperbolic plane one can easily construct a basis \( f_1, f_2 \) such that \( g(f_1, f_1) = g(f_2, f_2) = 0 \) and \( g(f_1, f_2) = 1 \). So that with respect to this basis \( g \) is given by the matrix

\[
\begin{bmatrix}
0 & 1 \\
1 & 0 
\end{bmatrix}
\]

This pair \( \{ f_1, f_2 \} \) is called a hyperbolic pair for \( E, g \). Now we return to \( \dim E \geq 2 \).

Lemma 7.35  There exists a subspace \( U \subset E \) such that \( E = \text{rad} E \perp U \) and \( U \) is nondegenerate.

**Proof.**  It is not too hard to see that \( \text{rad} U = \text{rad} U^{\perp} \). If \( \text{rad} U = 0 \) then \( \text{rad} U^{\perp} = 0 \) and vice versa. Now \( U + U^{\perp} \) is clearly direct since \( 0 = \text{rad} U = U \cap U^{\perp} \). Thus \( E = U \perp U^{\perp} \). \( \blacksquare \)

Lemma 7.36  Let \( g \) be nondegenerate and \( U \subset E \) some subspace. Suppose that \( U = \text{rad} U \perp W \) where \( \text{rad} W = 0 \). Then given a basis \( \{ u_1, \cdots, u_s \} \) for \( \text{rad} U \) there exists \( v_1, \cdots, v_s \in W^{\perp} \) such that each \( \{ u_i, v_i \} \) is a hyperbolic pair. Let \( P_1 = \text{span} \{ u_i, v_i \} \). Then

\[
E = W \perp P_1 \perp \cdots \perp P_s
\]

**Proof.**  Let \( W_1 = \text{span} \{ u_2, u_3, \cdots, u_s \} \oplus W \). Then \( W_1 \subseteq \text{rad} U \oplus W \) so (\( \text{rad} U \oplus W \))\(^{\perp} \) \( \supseteq W_1^{\perp} \). Let \( w_1 \in W_1^{\perp} \) but assume \( w_1 \notin (\text{rad} U \oplus W)\)\(^{\perp} \). Then we have \( g(u_1, w_1) \neq 0 \) so that \( P_1 = \text{span} \{ u_1, w_1 \} \) is a hyperbolic plane. Thus we can find \( v_1 \) such that \( u_1, v_1 \) is a hyperbolic pair for \( P_1 \). We also have

\[
U_1 = (u_2, u_3 \cdots u_s) \perp P_1 \perp W
\]

so we can proceed inductively since \( u_2, U_3, \cdots u_s \in \text{rad} U_1 \). \( \blacksquare \)

Definition 7.37  A subspace \( U \subset E \) is called totally isotropic if \( g|_U \equiv 0 \).
Proposition 7.38 (Witt decomposition) Suppose that $U \subset E$ is a maximal totally isotropic subspace and $e_1, e_2, \ldots, e_r$ a basis for $U$. Then there exist (null) vectors $f_1, f_2, \ldots, f_r$ such that each $\{e_i, f_i\}$ is a hyperbolic pair and $U' = \text{span}\{f_i\}$ is totally isotropic. Further

$$E = U \oplus U' \perp G$$

where $G = (U \oplus U')^\perp$.

Proof. Using the proof of the previous theorem we have $\text{rad} U = U$ and $W = 0$. The present theorem now follows. \hfill \blacksquare

Proposition 7.39 If $g$ is symmetric then $g|_G$ is definite.

Example 7.40 Let $E, g = \mathbb{C}^{2k}, g_0$ where

$$g_0(z, w) = \sum_{i=1}^{2k} z_i w_i$$

Let $\{e_1, \ldots, e_k, e_{k+1}, \ldots, e_{2k}\}$ be the standard basis of $\mathbb{C}^{2k}$. Define

$$\epsilon_j = \frac{1}{\sqrt{2}} (e_j + ie_{k+j}) \quad j = 1, \ldots, k$$

and

$$\eta_j = \frac{1}{\sqrt{2}} (e_i - ie_{k+j}).$$

Then letting $F = \text{span}\{\epsilon_i\}$, $F' = \text{span}\{\eta_i\}$ we have $\mathbb{C}^{2k} = F \oplus F'$ and $F$ is a maximally isotropic subspace. Also, each $\{\epsilon_j, \eta_j\}$ is a hyperbolic pair.

This is the most important example of a neutral space:

Proposition 7.41 A vector space $E$ with quadratic form is called neutral if the rank, that is, the dimension of a totally isotropic subspace, is $r = \dim E/2$. The resulting decomposition $F \oplus F'$ is called a (weak) polarization.

7.7.3 Witt’s Decomposition and Clifford Algebras

Even Dimension Suppose that $V, Q$ is quadratic space over $\mathbb{K}$. Let $\dim V = r$ and suppose that $V, Q$ is neutral. Then we have that $C_\mathbb{K}$ is isomorphic to $\text{End}(S)$ for an $r$ dimensional space $S$ (spinor space). In particular, $C_\mathbb{K}$ is a simple algebra.

Proof. Let $F \oplus F'$ be a polarization of $V$. Here, $F$ and $F'$ are maximal totally isotropic subspaces of $V$. Now let $\{x_1, \ldots, x_r, y_1, \ldots, y_r\}$ be a basis for $V$ such that $\{x_i\}$ is a basis for $F$ and $\{y_i\}$ a basis for $F'$. Set $f = y_1 y_2 \cdots y_h$. Now let $S$ be the span of elements of the form $x_{i_1} x_{i_2} \cdots x_{i_r} f$ where $1 \leq i_1 < \ldots < i_h \leq r$. $S$ is an ideal of $C_\mathbb{K}$ of dimension $2^r$. We define a representation $\rho$ of $C_\mathbb{K}$ in $S$ by

$$\rho(u) s = us$$
This can be shown to be irreducible so that we have the desired result.  ■

Now since we are interested in Spin which sits inside and in fact generates $C_0$ we need the following

**Proposition 7.42** $C_0$ is isomorphic to $\text{End}(S^+) \times \text{End}(S^-)$ where $S^+ = C_0 \cap S$ and $S^- = C_1 \cap S$.

This follows from the obvious fact that each of $C_0 f$ and $C_1 f$ are invariant under multiplication by $C_0$.

Now consider a real quadratic space $V, Q$ where $Q$ is positive definite. We have $\text{Spin}(n) \subset Cl^0(0) \subset C_0$ and $\text{Spin}(n)$ generates $C_0$. Thus the complex spin group representation of is just given by restriction and is semisimple factoring as $S^+ \oplus S^-$.  

Odd Dimension In the odd dimensional case we can not expect to find a polarization but this cloud turns out to have a silver lining. Let $x_0$ be a non-isotropic vector from $V$ and set $V_1 = (x_0)^\perp$. On $V_1$ we define a quadratic form $Q_1$ by

$$Q_1(y) = -Q(x_0)Q(y)$$

for $y \in V_1$. It can be shown that $Q_1$ is non-degenerate. Now notice that for $y \in V_1$ then $x_0 y = -yx_0$ and further

$$(x_0 y)^2 = -x_0^2 y^2 = -Q(x_0)Q(y) = Q_1(y)$$

so that by the universal mapping property the map

$$y \mapsto x_0 y$$

can be extended to an algebra morphism $h$ from $Cl(Q_1, V_1)$ to $C_K$. Now these two algebras have the same dimension and since $C_0$ is simple it must be an isomorphism. Now if $Q$ has rank $r$ then $Q_1, V_1$ is neutral and we obtain the following

**Theorem 7.43** If the dimension of $V$ is odd and $Q$ has rank $r$ then $C_0$ is represented irreducibly in a space $S^+$ of dimension $2^r$. In particular $C_0 \cong \text{End}(S^+)$.  

### 7.7.4 The Chirality operator

Let $V$ be a Euclidean vector space with associated positive definite quadratic form $Q$. Let $\{e_1, \ldots, e_n\}$ be an oriented orthonormal frame for $V$. We define the chirality operator $\tau$ to be multiplication in the associated (complexified) Clifford algebra by the element

$$\tau = (\sqrt{-1})^{n/2} e_1 \cdots e_n$$

if $n$ is even and by

$$\tau = (\sqrt{-1})^{(n+1)/2} e_1 \cdots e_n$$

if $n$ is odd.
if \( n \) is odd. Here \( \tau \in Cl(n) \) and does not depend on the choice of orthonormal oriented frame. We also have that \( \tau v = -v \tau \) for \( v \in V \) and \( \tau^2 = 1 \).

Let us consider the case of \( n \) even. Now we have seen that we can write \( V \otimes C = F \oplus \bar{F} \) where \( F \) is totally isotropic and of dimension \( n \). In fact we may assume that \( F \) has a basis \( \{ e_{2j-1} - ie_{2j} : 1 \leq j \leq n/2 \} \), where the \( e_i \) come from an oriented orthonormal basis. Let's use this polarization to once again construct the spin representation.

First note that \( Q \) (or its associated bilinear form) places \( F \) and \( \bar{F} \) in duality so that we can identify \( \bar{F} \) with the dual space \( F' \). Now set \( S = \wedge F \). First we show how \( V \) act on \( S \). Given \( v \in V \) consider \( v \in V \otimes C \) and decompose \( v = w + \bar{w} \) according to our decomposition above. Define \( \phi_w s = \sqrt{2} w \wedge s \) and \( \phi_{\bar{w}} s = -i(\bar{w}) s \).

where \( i \) is interior multiplication. Now extend \( \phi \) linearly to \( V \). Exercise Show that \( \phi \) extends to a representation of \( C \otimes Cl(n) \). Show that \( S^+ = \wedge^+ F \) is invariant under \( C_0 \). It turns out that \( \phi_\tau \) is \((-1)^k\) on \( \wedge^k F \)

7.7.5 Spin Bundles and Spin-c Bundles

7.7.6 Harmonic Spinors
Chapter 8

Classical Mechanics

Every body continues in its state of rest or uniform motion in a straight line, except insofar as it doesn’t.
Arthur Eddington, Sir

8.1 Particle motion and Lagrangian Systems

If we consider a single particle of mass \(m\) then Newton’s law is

\[
m \frac{d^2 \mathbf{x}}{dt^2} = \mathbf{F}(\mathbf{x}(t), t)
\]

The force \(\mathbf{F}\) is **conservative** if it doesn’t depend on time and there is a potential function \(V : \mathbb{R}^3 \rightarrow \mathbb{R}\) such that \(\mathbf{F}(\mathbf{x}) = -\nabla V(\mathbf{x})\). Assume this is the case.

Then Newton’s law becomes

\[
m \frac{d^2 \mathbf{x}(t)}{dt^2} + \nabla V(\mathbf{x}(t)) = 0.
\]

Consider the Affine space \(C(I, \mathbf{x}_1, \mathbf{x}_2)\) of all \(C^2\) paths from \(\mathbf{x}_1\) to \(\mathbf{x}_2\) in \(\mathbb{R}^3\) defined on the interval \(I = [t_1, t_2]\). This is an Affine space modeled on the Banach space \(C^r_0(I)\) of all \(C^r\) functions \(\varepsilon : I \rightarrow \mathbb{R}^3\) with \(\varepsilon(t_1) = \varepsilon(t_1) = 0\) and with the norm

\[
\|\varepsilon\| = \sup_{t \in I} \{|\varepsilon(t)| + |\varepsilon'(t)| + |\varepsilon''(t)|\}.
\]

If we define the fixed affine linear path \(\mathbf{a} : I \rightarrow \mathbb{R}^3\) by \(\mathbf{a}(t) = \mathbf{x}_1 + \frac{t-t_1}{t_2-t_1}(\mathbf{x}_2 - \mathbf{x}_1)\) then all we have a coordinatization of \(C(I, \mathbf{x}_1, \mathbf{x}_2)\) by \(C^r_0(I)\) given by the single chart \(\psi : \mathbf{c} \mapsto \mathbf{c} - \mathbf{a} \in C^r_0(I)\). Then the tangent space to \(C(I, \mathbf{x}_1, \mathbf{x}_2)\) at a fixed path \(c_0\) is just \(C^r_0(I)\). Now we have the function \(S\) defined on \(C(I, \mathbf{x}_1, \mathbf{x}_2)\) by

\[
S(\mathbf{c}) = \int_{t_1}^{t_2} \left( \frac{1}{2} m \|\mathbf{c}'(t)\|^2 - V(\mathbf{c}(t)) \right) dt
\]
The variation of \( S \) is just the 1-form \( \delta S : C^2_0(I) \to \mathbb{R} \) defined by

\[
\delta S \cdot \varepsilon = \left. \frac{d}{d\tau} \right|_{\tau=0} S(c_0 + \tau \varepsilon)
\]

Let us suppose that \( \delta S = 0 \) at \( c_0 \). Then we have

\[
0 = \left. \frac{d}{d\tau} \right|_{\tau=0} S(c_0'(t) + \tau \varepsilon)
\]

\[
= \left. \frac{d}{d\tau} \right|_{\tau=0} \int_{t_1}^{t_2} \left( \frac{1}{2} m \|c_0'(t) + \tau \varepsilon'(t)\|^2 - V(c_0(t) + \tau \varepsilon(t)) \right) dt
\]

\[
= \int_{t_1}^{t_2} \left( mc_0'(t) \cdot \frac{d}{dt} \varepsilon(t) - \frac{d}{dt} \varepsilon(t) \right) dt
\]

\[
= \int_{t_1}^{t_2} \left( mc_0''(t) - \nabla V(c_0(t)) \cdot \varepsilon(t) \right) dt
\]

Now since this is true for every choice of \( \varepsilon \in C^2_0(I) \) we see that

\[
mc_0''(t) - \nabla V(c_0(t)) = 0
\]

thus we see that \( c_0(t) = x(t) \) is a critical point in \( C(I, x_1, x_2) \), that is, a stationary path, if and only if ?? is satisfied.

### 8.1.1 Basic Variational Formalism for a Lagrangian

In general we consider a differentiable manifold \( Q \) as our state space and then a Lagrangian density function \( L \) is given on \( TQ \). For example we can take a potential function \( V : Q \to \mathbb{R} \), a Riemannian metric \( g \) on \( Q \) and define the action functional \( S \) on the space of smooth paths \( I \to Q \) beginning and ending at a fixed points \( p_1 \) and \( p_2 \) given by

\[
S(c) = \int_{t_1}^{t_2} L(c'(t)) dt = \int_{t_1}^{t_2} \frac{1}{2} m \langle c'(t), c'(t) \rangle - V(c(t)) dt
\]

The tangent space at a fixed \( c_0 \) is the Banach space \( \Gamma^2_0(c_0 TQ) \) of \( C^2 \) vector fields \( \varepsilon : I \to TQ \) along \( c_0 \) that vanish at \( t_1 \) and \( t_2 \). A curve with tangent \( \varepsilon \) at \( c_0 \) is just a variation \( v : (-\epsilon, \epsilon) \times I \to Q \) such that \( \varepsilon(t) = \left. \frac{d}{ds} \right|_{s=0} v(s, t) \) is the
8.1. PARTICLE MOTION AND LAGRANGIAN SYSTEMS

variation vector field. Then we have

$$\delta S \cdot \varepsilon = \frac{\partial}{\partial s} \bigg|_{s=0} \int_{t_1}^{t_2} L \left( \frac{\partial v}{\partial \dot{t}}(s, t) \right) dt$$

$$= \int_{t_1}^{t_2} \frac{\partial}{\partial s} \bigg|_{s=0} L \left( \frac{\partial v}{\partial \dot{t}}(s, t) \right) dt$$

$$= \text{etc.}$$

Let us examine this in the case of $Q = U \subset \mathbb{R}^n$. With $q = (q^1, ..., q^n)$ being (general curvilinear) coordinates on $U$ we have natural (tangent bundle chart) coordinates $q, \dot{q}$ on $TU = U \times \mathbb{R}^n$. Assume that the variation has the form $q(s, t) = q(t) + s \varepsilon(t)$. Then we have

$$\delta S \cdot \varepsilon = \frac{\partial}{\partial s} \bigg|_{s=0} \int_{t_1}^{t_2} L(q(s, t), \dot{q}(s, t)) dt$$

$$= \int_{t_1}^{t_2} \frac{\partial L}{\partial q}(q, \dot{q}(s, t)) \cdot \varepsilon + \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \cdot \varepsilon dt$$

$$= \int_{t_1}^{t_2} \frac{\partial L}{\partial q}(q, \dot{q}(s, t)) \cdot \varepsilon - \frac{d}{dt} \bigg|_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \cdot \varepsilon dt$$

and since $\varepsilon$ was arbitrary we get the Euler-Lagrange equations for the motion

$$\frac{\partial L}{\partial q}(q, \dot{q}) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) = 0$$

In general, a time-independent Lagrangian on a manifold $Q$ is a smooth function on the tangent bundle:

$$L : TQ \rightarrow Q$$

and the associated action functional is a map from the space of smooth curves $C^\infty([a, b], Q)$ defined for $c : [a, b] \rightarrow Q$ by

$$S_L(c) = \int_a^b L(\dot{c}(t)) dt$$

where $\dot{c} : [a, b] \rightarrow TQ$ is the canonical lift (velocity). A time dependent Lagrangian is a smooth map

$$L : \mathbb{R} \times TQ \rightarrow Q$$

where the first factor $\mathbb{R}$ is the time $t$, and once again we have the associated action functional $S_L(c) = \int_a^b L(t, \dot{c}(t)) dt$.

Let us limit ourselves initially to the time independent case.
Definition 8.1 A smooth variation of a curve $c : [a, b] \to Q$ is a smooth map $\nu : [a, b] \times (-\epsilon, \epsilon) \to Q$ for small $\epsilon$ such that $\nu(t, 0) = c(t)$. We call the variation a variation with fixed endpoints if $\nu(a, s) = c(a)$ and $\nu(b, s) = c(b)$ for all $s \in (-\epsilon, \epsilon)$. Now we have a family of curves $\nu_s = \nu(., s)$. The infinitesimal variation at $\nu_0$ is the vector field along $c$ defined by $V(t) = \frac{d\nu}{ds}(t, 0)$. This $V$ is called the variation vector field for the variation. The differential of the functional $\delta S_L$ (classically called the first variation) is defined as

$$\delta S_L(c) \cdot V = \frac{d}{ds} \bigg|_{s=0} S_L(\nu_s) = \frac{d}{ds} \bigg|_{s=0} \int_a^b L(\nu_s(t)) dt$$

Remark 8.2 Every smooth vector field along $c$ is the variational vector field coming from some variation of $c$ and for any other variation $\nu'$ with $V(t) = \frac{d\nu}{ds}(t, 0)$ the above computed quantity $\delta S_L(c) \cdot V$ will be the same.

At any rate, if $\delta S_L(c) \cdot V = 0$ for all variations vector fields $V$ along $c$ and vanishing at the endpoints then we write $\delta S_L(c) = 0$ and call $c$ critical (or stationary) for $L$.

Now consider the case where the image of $c$ lies in some coordinate chart $U, \psi = q^1, q^2, ..., q^n$ and denote by $TU, T\psi = (q^1, q^2, ..., q^n, \dot{q}^1, \dot{q}^2, ..., \dot{q}^n)$ the natural chart on $TU \subset TQ$. In other words, $T\psi(\xi) = (q^1 \circ \tau(\xi), q^2 \circ \tau(\xi), ..., q^n \circ \tau(\xi), dq^1(\xi), dq^2(\xi), ..., dq^n(\xi))$. Thus the curve has coordinates

$$(c, \dot{c}) = (q^1(t), q^2(t), ..., q^n(t), \dot{q}^1(t), \dot{q}^2(t), ..., \dot{q}^n(t))$$

where now the $\dot{q}^i(t)$ really are time derivatives. In this local situation we can choose our variation to have the form $\dot{q}^i(t) = s \dot{q}^i(t)$ for some functions $\dot{q}^i(t)$ vanishing at $a$ and $b$ and some parameter $s$ with respect to which we will differentiate. The lifted variation is $(\dot{q}(t) + s\dot{q}(t), \dot{q}(t) + s\dot{q}(t))$ which is the obvious abbreviation for a path in $T\psi(TU) \subset \mathbb{R}^n \times \mathbb{R}^n$. Now we have seen above that the path $c$ will be critical if

$$\frac{d}{ds} \bigg|_{s=0} \int L(q(t) + s\dot{q}(t), \dot{q}(t) + s\dot{q}(t)) dt = 0$$

for all such variations and the above calculations lead to the result that

$$\frac{\partial}{\partial \dot{q}} L(q(t), \dot{q}(t)) - \frac{d}{dt} \frac{\partial}{\partial \dot{q}} L(q(t), \dot{q}(t)) = 0 \quad \text{Euler-Lagrange}$$

for any $L$-critical path (with image in this chart). Here $n = \dim(Q)$.

It can be shown that even in the case that the image of $c$ does not lie in the domain of a chart that $c$ is $L$-critical path if it can be subdivided into sub-paths lying in charts and $L$-critical in each such chart.
8.1.2 Two examples of a Lagrangian

**Example 8.3** Suppose that we have a 1-form \( \theta \in \mathcal{X}^*(Q) \). A 1-form is just a map \( \theta : TQ \rightarrow \mathbb{R} \) that happens to be linear on each fiber \( T_pQ \). Thus we may examine the special case of \( L = \theta \). In canonical coordinates \((q, \dot{q})\) again,

\[
L = \theta = \sum a_i(q) dq^i
\]

for some functions \( a_i(q) \). An easy calculation shows that the Euler-Lagrange equations become

\[
\left( \frac{\partial a_i}{\partial q^k} - \frac{\partial a_k}{\partial q^i} \right) \dot{q}^i = 0
\]

but on the other hand

\[
d\theta = \frac{1}{2} \sum \left( \frac{\partial a_j}{\partial q^i} - \frac{\partial a_i}{\partial q^j} \right) dq^i \wedge dq^j
\]

and one can conclude that if \( c = (q^i(t)) \) is critical for \( L = \theta \) then for any vector field \( X \) defined on the image of \( c \) we have

\[
\frac{1}{2} \sum \left( \frac{\partial a_j}{\partial q^i}(q^i(t)) - \frac{\partial a_i}{\partial q^j}(q^i(t)) \right) \dot{q}^i(t) X^j
\]

or \( d\theta(\dot{c}(t), X) = 0 \). This can be written succinctly as

\[
\iota_{\dot{c}(t)} d\theta = 0.
\]

**Example 8.4** Now let us take the case of a Riemannian manifold \( M, g \) and let \( L(v) = \frac{1}{2} g(v, v) \). Thus the action functional is the “energy”

\[
S_g(c) = \int g(\dot{c}(t), \dot{c}(t)) dt
\]

In this case the critical paths are just geodesics.

8.2 Symmetry, Conservation and Noether’s Theorem

Let \( G \) be a Lie group acting on a smooth manifold \( M \).

\[
\lambda : G \times M \rightarrow M
\]

As usual we write \( g \cdot x \) for \( \lambda(g, x) \). We have a fundamental vector field \( \xi^g \) associated to every \( \xi \in \mathfrak{g} \) defined by the rule

\[
\xi^g(p) = T_{(e,p)}\lambda \cdot (., 0)
\]
or equivalently by the rule
\[
\xi^t(p) = \frac{d}{dt} \bigg|_0 \exp(t\xi) \cdot p
\]

The map \( \xi \mapsto \xi^t \) is a Lie algebra anti-homomorphism. Of course, here we are using the flow associated to \( \xi \)

\[
\varphi^\xi(t, p) := \varphi^\xi(t, p) = \exp(t\xi) \cdot p
\]

and it should be noted that \( t \mapsto \exp(t\xi) \) is the one parameter subgroup associated to \( \xi \) and to get the corresponding left invariant vector field \( X^\xi \in X^L(G) \)

we act on the right:

\[
X^\xi(g) = \frac{d}{dt} \bigg|_0 g \cdot \exp(t\xi)
\]

Now a diffeomorphism acts on a covariant \( k \)-tensor field contravariantly according to

\[
(\phi^*K)(v_1, \ldots, v_k) = K(\phi(p))(T\phi v_1, \ldots, T\phi v_k)
\]

Suppose that we are given a covariant tensor field \( \Upsilon \in T(M) \) on \( M \). We think of \( \Upsilon \) as defining some kind of extra structure on \( M \). The two main examples for our purposes are

1. \( \Upsilon = \langle \cdot, \cdot \rangle \) a nondegenerate covariant symmetric 2-tensor. Then \((M, \langle \cdot, \cdot \rangle)\) is a (semi-) Riemannian manifold.

2. \( \Upsilon = \omega \in \Omega^2(M) \) a non-degenerate 2-form. Then \((M, \omega)\) is a symplectic manifold.

Then \( G \) acts on \( \Upsilon \) since \( G \) acts on \( M \) as diffeomorphisms. We denote this natural (left) action by \( g \cdot \Upsilon \). If \( g \cdot \Upsilon = \Upsilon \) for all \( g \in G \) we say that \( G \) acts by symmetries of the pair \( M, \Upsilon \).

**Definition 8.5** In general, a vector field \( X \) on \((M, \Upsilon)\) is called an *infinitesimal symmetry* of the pair \((M, \Upsilon)\) if \( \mathcal{L}_X \Upsilon = 0 \). Other terminology is that \( X \) is a \( \Upsilon \)-*Killing field*. The usual notion of a Killing field in (pseudo-) Riemannian geometry is the case when \( \Upsilon = \langle \cdot, \cdot \rangle \) is the metric tensor.

**Example 8.6** A group \( G \) is called a symmetry group of a symplectic manifold \((M, \omega)\) if \( G \) acts by symplectomorphisms so that \( g \cdot \omega = \omega \) for all \( g \in G \). In this case, each \( \xi \in \mathfrak{g} \) is an infinitesimal symmetry of \( M, \omega \) meaning that

\[
\mathcal{L}_\xi \omega = 0
\]

where \( \mathcal{L}_\xi \) is by definition the same as \( \mathcal{L}_{\xi^t} \). This follows because if we let \( g_t = \exp(t\xi) \) then each \( g_t \) is a symmetry so \( g_t^* \omega = 0 \) and

\[
\mathcal{L}_\xi \omega = \frac{d}{dt} \bigg|_0 g_t^* \omega = 0
\]
8.2. SYMMETRY, CONSERVATION AND NOETHER’S THEOREM

8.2.1 Lagrangians with symmetries.

We need two definitions

**Definition 8.7** If $\phi : M \to M$ is a diffeomorphism then the induced tangent map $T\phi : TM \to TM$ is called the **canonical lift**.

**Definition 8.8** Given a vector field $X \in \mathfrak{X}(M)$ there is a lifting of $X$ to $\tilde{X} \in \mathfrak{X}(TM) = \Gamma(TM, TTM)$

$$
\tilde{X} : TM \to TTM \\
X : M \to TM
$$

such that the flow $\varphi^{\tilde{X}}$ is the canonical lift of $\varphi^X$.

$$
\varphi^{\tilde{X}} : TM \to TM \\
\varphi^X : M \to M
$$

In other words, $\varphi^{\tilde{X}} = T\varphi^X$. We simply define $\tilde{X}(v) = \frac{d}{dt}(T\varphi^X \cdot v)$.

**Definition 8.9** Let $\omega_L$ denote the unique 1-form on $Q$ that in canonical coordinates is $\omega_L = \sum a^i \frac{\partial L}{\partial \dot{q}^i} dq^i$.

**Theorem 8.10 (E. Noether)** If $X$ is an infinitesimal symmetry of the Lagrangian then the function $\omega_L(\tilde{X})$ is constant along any path $c : I \subset \mathbb{R}$ that is stationary for the action associated to $L$.

Let’s prove this using local coordinates $(q^i, \dot{q}^i)$ for $TU_\alpha \subset TQ$. It turn out that locally,

$$
\tilde{X} = \sum_i \left( a^i \frac{\partial}{\partial q^i} + \sum_j \frac{\partial a^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^j} \right)
$$

where $a^i$ is defined by $X = \sum a^i(q) \frac{\partial}{\partial q^i}$. Also, $\omega_L(\tilde{X}) = \sum a^i \frac{\partial L}{\partial \dot{q}^i}$. Now suppose that $q^i(t), \dot{q}^i(t) = \frac{d}{dt}q^i(t)$ satisfies the Euler-Lagrange equations. Then

$$
\frac{d}{dt} \omega_L(\tilde{X})(q^i(t), \dot{q}^i(t)) = \frac{d}{dt} \sum a^i(q^i(t)) \frac{\partial L}{\partial q^i}(q^i(t), \dot{q}^i(t))
$$

$$
= \sum \frac{da^i}{dt} (q^i(t)) \frac{\partial L}{\partial q^i}(q^i(t), \dot{q}^i(t)) + a^i(q^i(t)) \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i}(q^i(t), \dot{q}^i(t))
$$

$$
= \sum_i \left[ \sum_j \frac{da^i}{dt} \dot{q}^j(t) \frac{\partial L}{\partial q^j}(q^i(t), \dot{q}^i(t)) + a^i(q^i(t)) \frac{\partial L}{\partial \dot{q}^i}(q^i(t), \dot{q}^i(t)) \right]
$$

$$
= dL(X) = \mathcal{L}_X L = 0
$$
CHAPTER 8. CLASSICAL MECHANICS

This theorem tells us one case when we get a conservation law. A conservation law is a function $C$ on $TQ$ (or $T^{*}Q$ for the Hamiltonian flow) such that $C$ is constant along solution paths. (i.e. stationary for the action or satisfying the Euler-Lagrange equations.)

\[ L : TQ \to Q \]

let $X \in T(TQ)$.

8.2.2 Lie Groups and Left Invariants Lagrangians

Recall that $G$ act on itself by left translation $l_{g} : G \to G$. The action lifts to the tangent bundle $Tl_{g} : TG \to TG$. Suppose that $L : TG \to R$ is invariant under this left action so that $L(Tl_{g}X_{h}) = L(X_{h})$ for all $g, h \in G$. In particular, $L(Tl_{g}X_{e}) = L(X_{e})$ so $L$ is completely determined by its restriction to $T_{e}G = g$.

Define the restricted Lagrangian function by $\Lambda = L|_{T_{e}G}$. We view the differential $d\Lambda$ as a map $d\Lambda : g \to R$ and so in fact $d\lambda \in g^{*}$. Next, recall that for any $\xi \in g$ the map $ad_{\xi} : g \to g$ is defined by $ad_{\xi}v = [\xi, v]$ and we have the adjoint map $ad^{*}_{\xi} : g^{*} \to g^{*}$. Now let $t : g \to T_{c}(t)$ be a motion of the system and define the “body velocity” by $\nu_{c}(t) = Tl_{c}(t) \cdot c(t) = \omega_{G}(c'(t))$. Then we have

Theorem 8.11 Assume $L$ is invariant as above. The curve $c(\cdot)$ satisfies the Euler-Lagrange equations for $L$ if and only if

\[ \frac{d}{dt}d\Lambda(\nu_{c}(t)) = ad^{*}_{\nu_{c}(t)}d\Lambda \]

8.3 The Hamiltonian Formalism

Let us now examine the change to a description in cotangent chart $q, p$ so that for a covector at $q$ given by $a(q)\cdot dq$ has coordinates $q, a$. Our method of transfer to the cotangent side is via the Legendre transformation induced by $L$. In fact, this is just the fiber derivative defined above. We must assume that the map $F : (q, \dot{q}) \mapsto (q, p) = (q, \frac{\partial}{\partial \dot{q}}L(q, \dot{q}))$ is a diffeomorphism (this is written with respect to the two natural charts on $TU$ and $T^{*}U$). Again this just means that the Lagrangian is nondegenerate. Now if $v(t) = (q(t), \dot{q}(t))$ is (a lift of) a solution curve then defining the Hamiltonian function

\[ \tilde{H}(q, \dot{q}) = \frac{\partial}{\partial \dot{q}}L(q, \dot{q}) \cdot \dot{q} - L(q, \dot{q}) \]

we compute with $\dot{q} = \frac{d}{dt}q$

\[ \frac{d}{dt}\tilde{H}(q, \dot{q}) = \frac{d}{dt}\frac{\partial}{\partial \dot{q}}L(q, \dot{q}) \cdot \dot{q} - \frac{d}{dt}L(q, \dot{q}) \]

\[ = \frac{\partial}{\partial \dot{q}}L(q, \dot{q}) \cdot \frac{d}{dt}q \cdot \frac{d}{dt}\frac{\partial}{\partial \dot{q}}L(q, \dot{q}) \cdot \dot{q} - \frac{d}{dt}L(q, \dot{q}) \]

\[ = 0 \]
we have used that the Euler-Lagrange equations \( \frac{\partial L}{\partial q}(q, \dot{q}) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) = 0 \). Thus differential form \( d\tilde{H} = \frac{\partial \tilde{H}}{\partial q} dq + \frac{\partial \tilde{H}}{\partial \dot{q}} d\dot{q} \) is zero on the velocity \( v'(t) = \frac{d}{dt}(q, \dot{q}) \)

\[
\tilde{H} \cdot v'(t) = \frac{\partial \tilde{H}}{\partial q} \frac{dq}{dt} + \frac{\partial \tilde{H}}{\partial \dot{q}} \frac{d\dot{q}}{dt} = 0
\]

We then use the inverse of this diffeomorphism to transfer the Hamiltonian function to a function \( H(q, p) = F^{-1} \tilde{H}(q, \dot{q}) = p \cdot \dot{q}(q, p) - L(q, \dot{q}, q, p) \)

Now if \( q(t), \dot{q}(t) \) is a solution curve then its image \( b(t) = F \circ v(t) = (q(t), p(t)) \) satisfies

\[
dH(b'(t)) = (dH \cdot TF \cdot v'(t)) = (F^* dH) \cdot v'(t) = d(F^* H) \cdot v'(t) = d\tilde{H} \cdot v'(t) = 0
\]

so we have that

\[
0 = dH(b'(t)) = \frac{\partial H}{\partial q} \frac{dq}{dt} + \frac{\partial H}{\partial p} \frac{dp}{dt}
\]

but also

\[
\frac{\partial}{\partial p} H(q, p) = \dot{q} + p \frac{\partial \dot{q}}{\partial p} - \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial p} = \dot{q} = \frac{dq}{dt}
\]

solving these last two equations simultaneously we arrive at Hamilton’s equations of motion:

\[
\frac{dq}{dt}(t) = \frac{\partial H}{\partial p}(q(t), p(t))
\]

\[
\frac{dp}{dt}(t) = -\frac{\partial H}{\partial q}(q(t), p(t))
\]

or

\[
\frac{d}{dt} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}
\]

Remark 8.12 One can calculate directly that \( \frac{dH}{dt}(q(t), p(t)) = 0 \) for solutions these equations. If the Lagrangian was originally given by \( L = \frac{1}{2} K - V \) for some kinetic energy function and a potential energy function then this amounts to conservation of energy. We will see that this follows from a general principle below.
Chapter 9

Complex Manifolds

9.1 Some complex linear algebra

The set of all $n$-tuples of complex $\mathbb{C}^n$ numbers is a complex vector space and by choice of a basis, every complex vector space of finite dimension (over $\mathbb{C}$) is linearly isomorphic to $\mathbb{C}^n$ for some $n$. Now multiplication by $i := \sqrt{-1}$ is a complex linear map $\mathbb{C}^n \rightarrow \mathbb{C}^n$ and since $\mathbb{C}^n$ is also a real vector space $\mathbb{R}^{2n}$ under the identification

$$(x^1 + iy^1, ..., x^n + iy^n) \mapsto (x^1, y^1, ..., x^n, y^n)$$

we obtain multiplication by $i$ as a real linear map $J_0 : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ given by the matrix

$$
\begin{bmatrix}
0 & -1 \\
1 & 0 \\
0 & -1 \\
1 & 0 \\
\vdots \\
0 & -1 \\
1 & 0 \\
\end{bmatrix}.
$$

Conversely, if $V$ is a real vector space of dimension $2n$ and there is a map $J : V \rightarrow V$ with $J^2 = -1$ then we can define the structure of a complex vector space on $V$ by defining the scalar multiplication by complex numbers via the formula

$$(x + iy)v := xv + yJv \text{ for } v \in V.$$  

Denote this complex vector space by $V_J$. Now if $e_1, ..., e_n$ is a basis for $V_J$ (over $\mathbb{C}$) then we claim that $e_1, ..., e_n, Je_1, ..., Je_n$ is a basis for $V$ over $\mathbb{R}$. We only need to show that $e_1, ..., e_n, Je_1, ..., Je_n$ span. For this let $v \in V$. Then for some complex numbers $c_i = a^i + ib^i$ we have $v = \sum c_i e_i = \sum (a^i + ib^i) e_j = \sum a^i e_j + \sum b^i Je_j$ which is what we want.
Next we consider the complexification of $V$ which is $V_C := \mathbb{C} \otimes V$. Now any real basis $\{f_j\}$ of $V$ is also a basis for $V_C$ if we identify $f_j$ with $1 \otimes f_j$. Furthermore, the linear map $J : V \to V$ extends to a complex linear map $J : V_C \to V_C$ and still satisfies $J^2 = -1$. Thus this extension has eigenvalues $i$ and $-i$. Let $V^{1.0}$ be the eigenspace for $i$ and let $V^{0.1}$ be the $-i$ eigenspace. Of course we must have $V_C = V^{1.0} \oplus V^{0.1}$. The reader may check that the set of vectors $\{e_1 - iJ e_1, ..., e_n - iJ e_n\}$ span $V^{1.0}$ while $\{e_1 + iJ e_1, ..., e_n + iJ e_n\}$ span $V^{0.1}$. Thus we have a convenient basis for $V_C = V^{1.0} \oplus V^{0.1}$.

**Lemma 9.1** There is a natural complex linear isomorphism $V_J \cong V^{1.0}$ given by $e_i \mapsto e_i - iJ e_i$. Furthermore, the conjugation map on $V_C$ interchanges the spaces $V^{1.0}$ and $V^{0.1}$.

Let us apply these considerations to the simple case of the complex plane $\mathbb{C}$. The realification is $\mathbb{R}^2$ and the map $J$ is

$$
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
$$

If we identify the tangent space of $\mathbb{R}^{2n}$ at $0$ with $\mathbb{R}^{2n}$ itself then $\{ \frac{\partial}{\partial x^i} \big|_0, \frac{\partial}{\partial y^j} \big|_0 \}_{1 \leq i \leq n}$ is a basis for $\mathbb{R}^{2n}$. A complex basis for $\mathbb{R}_e^2 \cong \mathbb{C}$ is $e_1 = \frac{\partial}{\partial x} \big|_0$ and so $\{ \frac{\partial}{\partial x} \big|_0, J \frac{\partial}{\partial y} \big|_0 \}$ provides a basis for $\mathbb{R}^2$. This is clear anyway since $J \frac{\partial}{\partial x} \big|_0 = \frac{\partial}{\partial y} \big|_0$. Now the complexification of $\mathbb{R}^2$ is $\mathbb{R}^2_e$, which has basis consisting of $e_1 - iJ e_1 = \frac{\partial}{\partial x} \big|_0 - i \frac{\partial}{\partial y} \big|_0$ and $e_1 + iJ e_1 = \frac{\partial}{\partial x} \big|_0 + i \frac{\partial}{\partial y} \big|_0$. Multiplying these basis vectors by $1/2$ gives basis vectors which are usually denoted by $\frac{\partial}{\partial x} \big|_0$ and $\frac{\partial}{\partial y} \big|_0$. More generally, we see that if $\mathbb{C}^n$ is realified to $\mathbb{R}^{2n}$ which is then complexified to $\mathbb{R}_e^{2n} := \mathbb{C} \otimes \mathbb{R}^{2n}$ then a basis for $\mathbb{R}_e^{2n}$ is given by

$$
\left\{ \frac{\partial}{\partial z^1} \big|_0, \ldots, \frac{\partial}{\partial z^n} \big|_0, \frac{\partial}{\partial \overline{z}^1} \big|_0, \ldots, \frac{\partial}{\partial \overline{z}^n} \big|_0 \right\}
$$

where

$$
\frac{\partial}{\partial z^1} \big|_0 := \frac{1}{2} \left( \frac{\partial}{\partial x^1} \big|_0 - i \frac{\partial}{\partial y^1} \big|_0 \right)
$$

and

$$
\frac{\partial}{\partial \overline{z}^1} \big|_0 := \frac{1}{2} \left( \frac{\partial}{\partial x^1} \big|_0 + i \frac{\partial}{\partial y^1} \big|_0 \right).
$$

Now if we consider the tangent bundle $U \times \mathbb{R}^{2n}$ of an open set $U \subset \mathbb{R}^{2n}$ then we have the basis vector fields $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}$. We can complexify the tangent bundle of $U \times \mathbb{R}^{2n}$ to get $U \times \mathbb{R}_e^{2n}$ and then following the ideas above we have that the fields $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}$ also span each complexified tangent space $T_p U := \{p\} \times \mathbb{R}_e^{2n}$. On the other hand, so do the fields $\{ \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \overline{z}^1}, \ldots, \frac{\partial}{\partial \overline{z}^n} \}$. Now if $\mathbb{R}^{2n}$ had some complex structure, say $\mathbb{C}^n \cong (\mathbb{R}^{2n}, J_0)$, then $J_0$ defines a bundle map...
9.1. SOME COMPLEX LINEAR ALGEBRA

given \( J_0 : TU \to TU \) given by \((p, v) \mapsto (p, J_0 v)\). This can be extended to a complex bundle map \( J_0 : TU_\mathbb{C} = \mathbb{C} \otimes TU \to TU_\mathbb{C} = \mathbb{C} \otimes TU \) and we get a bundle decomposition

\[
TU_\mathbb{C} = T^{1,0}U \oplus T^{0,1}U
\]

where \( \frac{\partial}{\partial z^i}, \ldots, \frac{\partial}{\partial z^n} \) spans \( T^{1,0}U \) at each point and \( \frac{\partial}{\partial \bar{z}^i}, \ldots, \frac{\partial}{\partial \bar{z}^n} \) spans \( T^{0,1}U \).

Now the symbols \( \frac{\partial}{\partial z^i} \), etc., already have meaning as differential operators.

Let us now show that this view is at least consistent with what we have done above. For a smooth complex valued function \( f : U \subset \mathbb{C}^n \to \mathbb{C} \) we have for \( p = (z_1, \ldots, z_n) \in U \)

\[
\frac{\partial}{\partial z^i} \bigg|_p f = \frac{1}{2} \left( \frac{\partial}{\partial x^i} \bigg|_p f - i \frac{\partial}{\partial y^i} \bigg|_p f \right)
\]

\[
= \frac{1}{2} \left( \frac{\partial}{\partial x^i} \bigg|_p u + 2i \frac{\partial}{\partial y^i} \bigg|_p u - \frac{\partial}{\partial x^i} \bigg|_p iv - i \frac{\partial}{\partial y^i} \bigg|_p iv \right)
\]

\[
= \frac{1}{2} \left( \frac{\partial u}{\partial x^i} \bigg|_p + \frac{\partial v}{\partial y^i} \bigg|_p \right) + i \frac{1}{2} \left( \frac{\partial u}{\partial y^i} \bigg|_p - \frac{\partial v}{\partial x^i} \bigg|_p \right).
\]

and

\[
\frac{\partial}{\partial \bar{z}^i} \bigg|_p f = \frac{1}{2} \left( \frac{\partial}{\partial x^i} \bigg|_p f + i \frac{\partial}{\partial y^i} \bigg|_p f \right)
\]

\[
= \frac{1}{2} \left( \frac{\partial}{\partial x^i} \bigg|_p u - 2i \frac{\partial}{\partial y^i} \bigg|_p u + \frac{\partial}{\partial x^i} \bigg|_p iv + i \frac{\partial}{\partial y^i} \bigg|_p iv \right)
\]

\[
= \frac{1}{2} \left( \frac{\partial u}{\partial x^i} \bigg|_p - \frac{\partial v}{\partial y^i} \bigg|_p \right) + i \frac{1}{2} \left( \frac{\partial u}{\partial y^i} \bigg|_p + \frac{\partial v}{\partial x^i} \bigg|_p \right).
\]

**Definition 9.2** A function \( f : U \subset \mathbb{C}^n \to \mathbb{C} \) is called **holomorphic** if

\[
\frac{\partial}{\partial z^i} f \equiv 0 \quad (\text{all } i)
\]
on \( U \). A function \( f \) is called **antiholomorphic** if

\[
\frac{\partial}{\partial z^i} f \equiv 0 \quad (\text{all } i).
\]

**Definition 9.3** A map \( f : U \subset \mathbb{C}^n \to \mathbb{C}^m \) given by functions \( f_1, \ldots, f_m \) is called **holomorphic** (resp. **antiholomorphic**) if each component function \( f_1, \ldots, f_m \) is holomorphic (resp. antiholomorphic).

Now if \( f : U \subset \mathbb{C}^n \to \mathbb{C} \) is holomorphic then by definition \( \frac{\partial}{\partial z^i} \bigg|_p f \equiv 0 \) for all \( p \in U \) and so we have the **Cauchy-Riemann equations**

\[
\frac{\partial u}{\partial x^i} = \frac{\partial v}{\partial y^i} \quad \text{(Cauchy-Riemann)}
\]

\[
\frac{\partial v}{\partial x^i} = -\frac{\partial u}{\partial y^i}
\]
and from this we see that for holomorphic $f$

$$\frac{\partial f}{\partial z^i} = \frac{\partial u}{\partial x^i} + i \frac{\partial v}{\partial x^i} = \frac{\partial f}{\partial x^i}$$

which means that as derivations on the sheaf $\mathcal{O}$ of locally defined holomorphic functions on $\mathbb{C}^n$, the operators $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial x^i}$ are equal. This corresponds to the complex isomorphism $T^{1,0}U \cong (TU, J_0)$ which comes from the isomorphism in lemma ???. In fact, if one looks at a function $f: \mathbb{R}^{2n} \to \mathbb{C}$ as a differentiable map of real manifolds then with $J_0$ giving the isomorphism $\mathbb{R}^{2n} \cong \mathbb{C}^n$, our map $f$ is holomorphic if and only if

$$Tf \circ J_0 = J_0 \circ Tf$$

or in other words

$$\begin{pmatrix}
\frac{\partial u}{\partial x^1} & \frac{\partial u}{\partial x^2} & \cdots \\
\frac{\partial v}{\partial x^1} & \frac{\partial v}{\partial x^2} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 1 \\
0 & 0 & \ddots
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial u}{\partial x^1} & \frac{\partial u}{\partial x^2} & \cdots \\
\frac{\partial v}{\partial x^1} & \frac{\partial v}{\partial x^2} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}.$$

This last matrix equation is just the Cauchy-Riemann equations again.

### 9.2 Complex structure

**Definition 9.4** A manifold $M$ is said to be an **almost complex manifold** if there is a smooth bundle map $J: TM \to TM$, called an **almost complex structure**, having the property that $J^2 = -I$.

**Definition 9.5** A **complex manifold** $M$ is a manifold modeled on $\mathbb{C}^n$ for some $n$, together with an atlas for $M$ such that the transition functions are all holomorphic maps. The charts from this atlas are called **holomorphic charts**. We also use the phrase “holomorphic coordinates”.

Since as real normed vector spaces we have $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and since homomorphic maps are always smooth we see that underlying every complex manifold of dimension $n$ there is a real manifold of dimension $2n$. Furthermore, we will see that each (real) tangent space of the underlying real manifold of a complex manifold has a natural almost complex structure. Those almost complex structures arising in this way are also called complex structures. Not all almost complex structures arise in this way (there is an integrability condition).

**Example 9.6** Let $S^2(1/2) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1/4\}$ be given coordinates $\psi^+: (x_1, x_2, x_3) \mapsto \frac{1}{1-x_3}(x_1 + ix_2) \in \mathbb{C}$ on $U^+: = \{(x_1, x_2, x_3) \in$
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\( S^2 : 1 - x_3 \neq 0 \) and \( \psi^- : (x_1, x_2, x_3) \mapsto \frac{1}{x_1 + x_2^2} (x_1 + ix_2) \in \mathbb{C} \) on \( U^- := \{(x_1, x_2, x_3) \in S^2 : 1 + x_3 \neq 0 \} \). The chart overlap map or transition function is \( \psi^- \circ \psi^+(z) = 1/z \). Since on \( \psi^+U^+ \cap \psi^-U^- \) the map \( z \mapsto 1/z \) is a biholomorphism we see that \( S^2(1/2) \) can be given the structure of a complex 1-manifold.

Another way to get the same complex 1-manifold is given by taking two copies of the complex plane, say \( \mathbb{C}_z \) with coordinate \( z \) and \( \mathbb{C}_w \) with coordinate \( w \) and then identify \( \mathbb{C}_z \) with \( \mathbb{C}_w - \{0\} \) via the map \( w = 1/z \). This complex surface is of course topologically a sphere and is also the 1 point compactification of the complex plane. As the reader will no doubt already be aware, this complex 1-manifold is called the Riemann sphere.

Example 9.7 Let \( P_n(\mathbb{C}) \) be the set of all complex lines through the origin in \( \mathbb{C}^{n+1} \), which is to say, the set of all equivalence classes of nonzero elements of \( \mathbb{C}^{n+1} \) under the equivalence relation

\[
(z^1, ..., z^{n+1}) \sim \lambda(z^1, ..., z^{n+1}) \text{ for } \lambda \in \mathbb{C}
\]

For each \( i \) with \( 1 \leq i \leq n + 1 \) define the set

\[
U_i := \{[z^1, ..., z^{n+1}] \in P_n(\mathbb{C}) : z^i \neq 0\}
\]

and corresponding map \( \psi_i : U_i \to \mathbb{C}^n \) by

\[
\psi_i([z^1, ..., z^{n+1}]) = \frac{1}{z^i}(z^1, ..., \hat{z}^i, ..., z^{n+1}) \in \mathbb{C}^n.
\]

One can check that these maps provide a holomorphic atlas for \( P_n(\mathbb{C}) \) which is therefore a complex manifold (complex projective \( n \)-space).

Example 9.8 Let \( M_{m \times n}(\mathbb{C}) \) be the space of \( m \times n \) complex matrices. This is clearly a complex manifold since we can always “line up” the entries to get a map \( M_{m \times n}(\mathbb{C}) \to \mathbb{C}^{mn} \) and so as complex manifolds \( M_{m \times n}(\mathbb{C}) \cong \mathbb{C}^{mn} \). A little less trivially we have the complex general linear group \( GL(n, \mathbb{C}) \) which is an open subset of \( M_{m \times n}(\mathbb{C}) \) and so is an \( n^2 \) dimensional complex manifold.

Example 9.9 (Grassmann manifold) To describe this important example we start with the set \( (M_{n \times k}(\mathbb{C}))_\sim \) of \( n \times k \) matrices with rank \( k < n \) (maximal rank). The columns of each matrix from \( (M_{n \times k}(\mathbb{C}))_\sim \) span a \( k \)-dimensional subspace of \( \mathbb{C}^n \). Define two matrices from \( (M_{n \times k}(\mathbb{C}))_\sim \) to be equivalent if they span the same \( k \)-dimensional subspace. Thus the set \( G(k, n) \) of equivalence classes is in one to one correspondence with the set of complex \( k \) dimensional subspaces of \( \mathbb{C}^n \). Now let \( U \) be the set of all \( [A] \in G(k, n) \) such that \( A \) has its first \( k \) rows linearly independent. This property is independent of the representative \( A \) of the equivalence class \( [A] \) and so \( U \) is a well defined set. This last fact is easily proven by a Gaussian reduction argument. Now every element \( [A] \in U \subset G(k, n) \) is an equivalence class that has a unique member \( A_0 \) of the form

\[
\begin{pmatrix}
I_{k \times k} \\
Z
\end{pmatrix}
\]


Thus we have a map on $U$ defined by $\Psi : [A] \mapsto Z \in M_{n-k \times k}(\mathbb{C}) \cong \mathbb{C}^{k(n-k)}$. We wish to cover $G(k,n)$ with sets $U_\sigma$ similar to $U$ and defined similar maps. Let $\sigma_{i_1 \ldots i_k}$ be the shuffle permutation that puts the $k$ columns indexed by $i_1, \ldots, i_k$ into the positions $1, \ldots, k$ without changing the relative order of the remaining columns. Now consider the set $U_{i_1 \ldots i_k}$ of all $[A] \in G(k,n)$ such that any representative $A$ has its $k$ rows indexed by $i_1, \ldots, i_k$ linearly independent. The permutation induces an obvious 1-1 onto map $\Psi_{i_1 \ldots i_k}$ from $U_{i_1 \ldots i_k}$ onto $U = U_{1 \ldots k}$. We now have maps $\Psi_{i_1 \ldots i_k} : U_{i_1 \ldots i_k} \to M_{n-k \times k}(\mathbb{C}) \cong \mathbb{C}^{k(n-k)}$ given by composition $\Psi_{i_1 \ldots i_k} := \Psi \circ \sigma_{i_1 \ldots i_k}$. These maps form an atlas $\{\Psi_{i_1 \ldots i_k} : U_{i_1 \ldots i_k}\}$ for $G(k,n)$ that turns out to be a holomorphic atlas (biholomorphic transition maps) and so gives $G(k,n)$ the structure of a complex manifold called the Grassmann manifold of complex $k$-planes in $\mathbb{C}^n$.

**Definition 9.10** A complex 1-manifold (so real dimension is 2) is called a Riemann surface.

If $S$ is a subset of a complex manifold $M$ such that near each $p_0 \in S$ there exists a holomorphic chart $U, \psi = (z^1, \ldots, z^n)$ such that $0 \in S \cap U$ if and only if $z^{k+1}(p) = \cdots = z^n(p) = 0$ then the coordinates $z^1, \ldots, z^k$ restricted to $U \cap S$ give a chart on the set $S$ and the set of all such charts gives $S$ the structure of a complex manifold. In this case we call $S$ a complex submanifold of $M$.

**Definition 9.11** In the same way as we defined differentiability for real manifolds we define the notion of a holomorphic map (resp. antiholomorphic map) from one complex manifold to another. Note however, that we must use holomorphic charts for the definition.

The proof of the following lemma is straightforward.

**Lemma 9.12** Let $z : U \to \mathbb{C}^n$ be a holomorphic chart with $p \in U$. Writing $z = (z^1, \ldots, z^n)$ and $z^k = x^k + iy^k$ we have that the map $J_p : T_pM \to T_pM$ given by

$$J_p \frac{\partial}{\partial x^i} \bigg|_p = \frac{\partial}{\partial y^i} \bigg|_p$$

$$J_p \frac{\partial}{\partial y^i} \bigg|_p = -\frac{\partial}{\partial x^i} \bigg|_p$$

is well defined independently of the choice of coordinates.

The maps $J_p$ combine to give a bundle map $J : TM \to TM$ and so we get an almost complex structure on $M$ called the almost complex structure induced by the holomorphic atlas.

**Definition 9.13** An almost complex structure $J$ on $M$ is said to be integrable if there is an holomorphic atlas giving the map $J$ as the induced almost complex structure. That is, if there is a family of admissible charts $x_\alpha : U_\alpha \to \mathbb{R}^{2n}$ such that after identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ the charts form a holomorphic atlas with $J$ the induced almost complex structure. In this case, we call $J$ a complex structure.
9.3 Complex Tangent Structures

Let $F_p(C)$ denote the algebra of germs of complex valued smooth functions at $p$ on a complex $n$-manifold $M$ thought of as a smooth real $2n$-manifold with real tangent bundle $TM$. Let $\text{Der}_p(F)$ be the space of derivations this algebra. It is not hard to see that this space is isomorphic to the complexified tangent space $T_pM_C = C \otimes T_pM$. The (complex) algebra of germs of holomorphic functions at a point $p$ in a complex manifold is denoted $O_p$ and the set of derivations of this algebra denoted $\text{Der}_p(O)$. We also have the algebra of germs of antiholomorphic functions at $p$ which is $\overline{O}_p$ and also $\text{Der}_p(\overline{O})$, the derivations of this algebra.

If $\psi : U \rightarrow C^n$ is a holomorphic chart then writing $\psi = (z^1, ..., z^n)$ and $z^k = x^k + iy^k$ we have the differential operators at $p \in U$:

$$\left\{ \frac{\partial}{\partial z^i} \bigg|_p, \frac{\partial}{\partial \bar{z}^i} \bigg|_p \right\}$$

(now transferred to the manifold). To be pedantic about it, we now denote the standard complex coordinates on $C^n$ by $w^i = u^i + iv^i$ and then

$$\frac{\partial}{\partial z^i} \bigg|_p f := \frac{\partial f \circ \psi^{-1}}{\partial w^i} \bigg|_{\psi(p)}$$

$$\frac{\partial}{\partial \bar{z}^i} \bigg|_p f := \frac{\partial f \circ \psi^{-1}}{\partial \bar{w}^i} \bigg|_{\psi(p)}$$

Thought of as derivations, these span $\text{Der}_p(F)$ but we have also seen that they span the complexified tangent space at $p$. In fact, we have the following:

$$T_pM_C = \text{span}_C \left\{ \frac{\partial}{\partial z^i} \bigg|_p, \frac{\partial}{\partial \bar{z}^i} \bigg|_p \right\} = \text{Der}_p(F)$$

$$T_pM^{1,0} = \text{span}_C \left\{ \frac{\partial}{\partial z^i} \bigg|_p \right\}$$

$$= \{ v \in \text{Der}_p(F) : vf = 0 \text{ for all } f \in \overline{O}_p \}$$

$$T_pM^{0,1} = \text{span}_C \left\{ \frac{\partial}{\partial \bar{z}^i} \bigg|_p \right\}$$

$$= \{ v \in \text{Der}_p(F) : vf = 0 \text{ for all } f \in O_p \}$$

and of course

$$T_pM = \text{span}_R \left\{ \frac{\partial}{\partial x^i} \bigg|_p, \frac{\partial}{\partial y^i} \bigg|_p \right\}.$$
$T_pM^{1,0}$ and $T_pM^{0,1}$ are independent of the holomorphic coordinates since we also have

$$T_p^{1,0}M = \ker\{J_p : T_pM \to T_pM\}$$

9.4 The holomorphic tangent map.

We leave it to the reader to verify that the constructions that we have at each tangent space globalize to give natural vector bundles $TM, TM^{1,0}$ and $TM^{0,1}$ (all with $M$ as base space).

Let $M$ and $N$ be complex manifolds and let $f : M \to N$ be a smooth map. The tangent map (on the underlying real manifolds) extends to a map of the complexified bundles $T_f : TM \to TN$. Now $TM = TM^{1,0} \oplus TM^{0,1}$ and similarly $TN = TN^{1,0} \oplus TN^{0,1}$. If $f$ is holomorphic then $T_f(T_p^{1,0}M) \subset T_f(p)^{1,0}N$. In fact, it is easily verified that the statement that $T_f(T_p^{1,0}M) \subset T_f(p)^{1,0}N$ is equivalent to the statement that the Cauchy-Riemann equations are satisfied by the local representative of $f$ in any holomorphic chart. As a result we have

**Proposition 9.14** $T_f(T_p^{1,0}M) \subset T_f(p)^{1,0}N$ for all $p$ if and only if $f$ is a holomorphic map.

The map given by the restriction $T_pf : T_p^{1,0}M \to T_f(p)^{1,0}N$ is called the **holomorphic tangent** map at $p$. Of course, these maps combine to give a bundle map.

9.5 Dual spaces

Let $M$ be a complex manifold and $J$ the induced complex structure map $TM \to TM$. The dual of $T_pM_C$ is $T_p^*M_C = C \otimes T^*_pM$. Now the map $J$ has a dual bundle map $J^* : T^*M_C \to T^*M_C$ that must also satisfy $J^* \circ J^* = -1$ and so we have at each $p \in M$, a decomposition by eigenspaces

$$T_p^*M_C = T_p^*M^{1,0} \oplus T_p^*M^{0,1}$$

corresponding to the eigenvalues $\pm i$.

**Definition 9.15** The space $T_p^*M^{1,0}$ is called the space of holomorphic covectors at $p$ while $T_p^*M^{0,1}$ is the space of antiholomorphic covectors at $p$.

We now choose a holomorphic chart $\psi : U \to C^n$ at $p$. Writing $\psi = (z^1, ..., z^n)$ and $z^k = x^k + iy^k$ we have the 1-forms

$$dz^k = dx^k + idy^k$$

and

$$dz^k = dx^k - idy^k.$$
9.5. DUAL SPACES

Equivalently, the pointwise definitions are \( dz_k \big|_p = dx_k \big|_p + i dy_k \big|_p \) and \( d\bar{z}_k \big|_p = dx_k \big|_p - i dy_k \big|_p \). Notice that we have the expected relations:

\[
dz_k \left( \frac{\partial}{\partial z^i} \right) = (dx_k + i dy_k) \left( \frac{1}{2} \frac{\partial}{\partial x^j} - \frac{1}{2} \frac{\partial}{\partial y^j} \right) = \frac{1}{2} \delta^k_j + \frac{1}{2} \delta^k_j = \delta^k_j \]

and similarly

\[
dz_k \left( \frac{\partial}{\partial \bar{z}^i} \right) = \delta^k_j \text{ and } d\bar{z}_k \left( \frac{\partial}{\partial z^i} \right) = \delta^k_j .
\]

Let us check the action of \( J^* \) on these forms:

\[
J^*(dz_k) \left( \frac{\partial}{\partial z^i} \right) = J^*(dx_k + i dy_k) \left( \frac{\partial}{\partial z^i} \right) = (dx_k + i dy_k) \left( J \frac{\partial}{\partial z^i} \right) = i(dx_k + i dy_k) \frac{\partial}{\partial z^i} = idz_k \left( \frac{\partial}{\partial z^i} \right)
\]

and

\[
J^*(dz_k) \left( \frac{\partial}{\partial \bar{z}^i} \right) = dz_k \left( J \frac{\partial}{\partial \bar{z}^i} \right) = -idz_k \left( \frac{\partial}{\partial \bar{z}^i} \right) = 0 = idz_k \left( \frac{\partial}{\partial \bar{z}^i} \right).
\]

We conclude that \( dz_k \big|_p \in T^*_p M^{1,0} \). A similar calculation shows that \( d\bar{z}_k \big|_p \in T^*_p M^{0,1} \) and in fact

\[
T^*_p M^{1,0} = \text{span} \left\{ dz_k \big|_p : k = 1, \ldots, n \right\}
\]

\[
T^*_p M^{0,1} = \text{span} \left\{ d\bar{z}_k \big|_p : k = 1, \ldots, n \right\}
\]

and \( \{ dz_1 \big|_p, \ldots, dz_n \big|_p, d\bar{z}_1 \big|_p, \ldots, d\bar{z}_n \big|_p \} \) is a basis for \( T^*_p M_{\mathbb{C}} \).

**Remark 9.16** If we don’t specify base points then we are talking about fields (over some open set) that form a basis for each fiber separately. As before these are called frame fields (e.g. \( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i} \)) or coframe fields (e.g. \( dz^k, d\bar{z}^k \)).
9.6 Examples
Under construction ASP misspelled on poipois

9.7 The holomorphic inverse and implicit functions theorems.

Let \((z^1, \ldots, z^n)\) and \((w^1, \ldots, w^m)\) be local coordinates on complex manifolds \(M\) and \(N\) respectively. Consider a smooth map \(f : M \rightarrow N\). We suppose that \(p \in M\) is in the domain of \((z^1, \ldots, z^n)\) and that \(q = f(p)\) is in the domain of the coordinates \((w^1, \ldots, w^m)\). Writing \(z^i = x^i + iy^i\) and \(w^i = u^i + iv^i\) we have the following Jacobian matrices:

1. If we consider the underlying real structures then we have the Jacobian given in terms of the frame \(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\) and \(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial v^i}\):

   \[
   J_p(f) = \begin{bmatrix}
   \frac{\partial u^1}{\partial x^1}(p) & \frac{\partial u^1}{\partial y^1}(p) & \frac{\partial u^1}{\partial x^2}(p) & \frac{\partial u^1}{\partial y^2}(p) & \cdots \\
   \frac{\partial v^1}{\partial x^1}(p) & \frac{\partial v^1}{\partial y^1}(p) & \frac{\partial v^1}{\partial x^2}(p) & \frac{\partial v^1}{\partial y^2}(p) & \cdots \\
   \frac{\partial u^2}{\partial x^1}(p) & \frac{\partial u^2}{\partial y^1}(p) & \frac{\partial u^2}{\partial x^2}(p) & \frac{\partial u^2}{\partial y^2}(p) & \cdots \\
   \frac{\partial v^2}{\partial x^1}(p) & \frac{\partial v^2}{\partial y^1}(p) & \frac{\partial v^2}{\partial x^2}(p) & \frac{\partial v^2}{\partial y^2}(p) & \cdots \\
   \vdots & \vdots & \vdots & \vdots & \ddots
   \end{bmatrix}
   \]

2. With respect to the bases \(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}\) and \(\frac{\partial}{\partial w^i}, \frac{\partial}{\partial \bar{w}^i}\) we have

   \[
   J_{p,\mathbb{C}}(f) = \begin{bmatrix}
   J_{11} & J_{12} & \cdots \\
   J_{12} & J_{22} & \cdots \\
   \vdots & \vdots & \ddots
   \end{bmatrix}
   \]

where the \(J_{ij}\) are blocks of the form

\[
\begin{bmatrix}
  \frac{\partial w^i}{\partial z^j} & \frac{\partial w^i}{\partial \bar{z}^j} \\
  \frac{\partial w^i}{\partial \bar{z}^j} & \frac{\partial w^i}{\partial z^j}
\end{bmatrix}.
\]

If \(f\) is holomorphic then these block reduce to the form

\[
\begin{bmatrix}
  \frac{\partial w^i}{\partial z^j} & 0 \\
  0 & \frac{\partial w^i}{\partial \bar{z}^j}
\end{bmatrix}.
\]

It is convenient to put the frame fields in the order \(\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial \bar{z}^n}\) and similarly for \(\frac{\partial}{\partial w^i}, \frac{\partial}{\partial \bar{w}^i}\). In this case we have for holomorphic \(f\)

\[
J_{p,\mathbb{C}}(f) = \begin{bmatrix}
  J^{1,0}_{1,1} & 0 \\
  0 & J^{1,0}_{1,3}
\end{bmatrix}
\]
9.7. THE HOLOMORPHIC INVERSE AND IMPLICIT FUNCTIONS THEOREMS

where

\[ J^{1,0}(f) = \begin{bmatrix} \frac{\partial w^1}{\partial z^1} \\ \vdots \\ \frac{\partial w^k}{\partial z^1} \end{bmatrix}, \]

\[ J^{1,0}(f) = \begin{bmatrix} \frac{\partial w^1}{\partial \overline{z}^1} \\ \vdots \\ \frac{\partial w^k}{\partial \overline{z}^1} \end{bmatrix}. \]

We shall call a basis arising from a holomorphic coordinate system “separated” when arranged this way. Note that \( J^{1,0} \) is just the Jacobian of the holomorphic tangent map \( T^{1,0} f : T^{1,0} M \to T^{1,0} N \) with respect to this the \textbf{holomorphic frame} \( \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n} \).

We can now formulate the following version of the inverse mapping theorem:

**Theorem 9.17** (1) Let \( U \) and \( V \) be open sets in \( \mathbb{C}^n \) and suppose that the map \( f : U \to V \) is holomorphic with \( J^{1,0}(f) \) nonsingular at \( p \in U \). Then there exists an open set \( U_0 \subset U \) containing \( p \) such that \( f|_{U_0} : U_0 \to f(U_0) \) is a 1-1 holomorphic map with holomorphic inverse. That is, \( f|_{U_0} \) is \textbf{biholomorphic}.

(2) Similarly, if \( f : U \to V \) is a holomorphic map between open sets of complex manifolds \( M \) and \( N \) then if \( T^{1,0}_p f : T^{1,0}_p M \to T^{1,0}_q N \) is a linear isomorphism then \( f \) is a biholomorphic map when restricted to a possibly smaller open set containing \( p \).

We also have a holomorphic version of the implicit mapping theorem.

**Theorem 9.18** (1) Let \( f : U \subset \mathbb{C}^n \to V \subset \mathbb{C}^k \) and let the component functions of \( f \) be \( f_1, \ldots, f_k \). If \( J^{1,0}_p(f) \) has rank \( k \) then there are holomorphic functions \( g^1, g^2, \ldots, g^k \) defined near \( 0 \in \mathbb{C}^{n-k} \) such that

\[ f(z^1, \ldots, z^n) = p \]

\[ \iff \]

\[ z^j = g^j(z^{k+1}, \ldots, z^n) \quad \text{for} \quad j = 1, \ldots, k \]

(2) If \( f : M \to N \) is a holomorphic map of complex manifolds and if for fixed \( q \in N \) we have that each \( p \in f^{-1}(q) \) is regular in the sense that \( T^{1,0}_p f : T^{1,0}_p M \to T^{1,0}_q N \) is surjective, then \( S := f^{-1}(q) \) is a complex submanifold of (complex) dimension \( n-k \).

**Example 9.19** The map \( \varphi : \mathbb{C}^{n+1} \to \mathbb{C} \) given by \( (z^1, \ldots, z^{n+1}) \mapsto (z^1)^2 + \cdots + (z^{n+1})^2 \) has Jacobian at any \((z^1, \ldots, z^{n+1}) \) given by

\[ \begin{bmatrix} 2z^1 & 2z^2 & \cdots & 2z^{n+1} \end{bmatrix} \]

which has rank 1 as long as \((z^1, \ldots, z^{n+1}) \neq 0\). Thus \( \varphi^{-1}(1) \) is a complex submanifold of \( \mathbb{C}^{n+1} \) having complex dimension \( n \). **Warning:** This manifold is not the same as the sphere given by \(|z^1|^2 + \cdots + |z^{n+1}|^2 = 1\) which is a real submanifold of \( \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2} \) of real dimension \( 2n+1 \).
Chapter 10
Symplectic Geometry

Equations are more important to me, because politics is for the present, but an equation is something for eternity
   - Einstein

10.1 Symplectic Linear Algebra

A (real) symplectic vector space is a pair $V, \alpha$ where $V$ is a (real) vector space and $\alpha$ is a nondegenerate alternating (skew-symmetric) bilinear form $\alpha : V \times V \to \mathbb{R}$. The basic example is $\mathbb{R}^{2n}$ with

$\alpha_0(x, y) = x^t J_n y$

where

$$J_n = \begin{pmatrix} 0 & I_{n \times n} \\ -I_{n \times n} & 0 \end{pmatrix}.$$

The standard symplectic form on $\alpha_0$ is typical. It is a standard fact from linear algebra that for any $N$ dimensional symplectic vector space $V, \alpha$ there is a basis $e_1, ..., e_n, f^1, ..., f^n$ called a symplectic basis such that the matrix that represents $\alpha$ with respect to this basis is the matrix $J_n$. Thus we may write

$$\alpha = e^1 \wedge f_1 + ... + e^n \wedge f_n$$

where $e^1, ..., e^n, f_1, ..., f_n$ is the dual basis to $e_1, ..., e_n, f^1, ..., f^n$. If $V, \eta$ is a vector space with a not necessarily nondegenerate alternating form $\eta$ then we can define the null space

$$N_\eta = \{ v \in V : \eta(v, w) = 0 \text{ for all } w \in V \}.$$

On the quotient space $\overline{V} = V/N_\eta$ we may define $\overline{\eta}(\overline{v}, \overline{w}) = \eta(v, w)$ where $v$ and $w$ represent the elements $\overline{v}, \overline{w} \in \overline{V}$. Then $\overline{V}, \overline{\eta}$ is a symplectic vector space called the symplectic reduction of $V, \eta$.
Proposition 10.1 For any $\eta \in \wedge V^*$ (regarded as a bilinear form) there is linearly independent set of elements $e^1, \ldots, e^k, f_1, \ldots, f_k$ from $V^*$ such that

$$\eta = e^1 \wedge f_1 + \ldots + e^k \wedge f_k$$

where $\dim(V) - 2k \geq 0$ is the dimension of $N_\eta$.

Definition 10.2 Note: The number $k$ is called the rank of $\eta$. The matrix that represents $\eta$ actually has rank $2k$ and so some might call $k$ the half rank of $\eta$.

Proof. Consider the symplectic reduction $\nabla, \eta$ of $V, \eta$ and choose set of elements $e^1, \ldots, e^k, f_1, \ldots, f_k$ such that $e^1, \ldots, e^k, f_1, \ldots, f_k$ form a symplectic basis of $\nabla, \eta$. Add to this set a basis $b_1, \ldots, b_l$ a basis for $N_\eta$ and verify that $e^1, \ldots, e^k, f_1, \ldots, f_k, b_1, \ldots, b_l$ must be a basis for $V$. Taking the dual basis one can check that

$$\eta = e^1 \wedge f_1 + \ldots + e^k \wedge f_k$$

by testing on the basis $e^1, \ldots, e^k, f_1, \ldots, f_k, b_1, \ldots, b_l$.

Now if $W$ is a subspace of a symplectic vector space then we may define

$$W^\perp = \{v \in V : \eta(v, w) = 0 \text{ for all } w \in W\}$$

and it is true that $\dim(W) + \dim(W^\perp) = \dim(V)$ but it is not necessarily the case that $W \cap W^\perp = 0$. In fact, we classify subspaces $W$ by two numbers: $d = \dim(W)$ and $\nu = \dim(W \cap W^\perp)$. If $\nu = 0$ then $\eta|_W : W$ is a symplectic space and so we call $W$ a symplectic subspace. At the opposite extreme, if $\nu = d$ then $W$ is called a Lagrangian subspace. If $W \subset W^\perp$ we say that $W$ is an isotropic subspace.

A linear transformation between symplectic vector spaces $\ell : V_1, \eta_1 \rightarrow V_2, \eta_2$ is called a symplectic linear map if $\eta_2(\ell(v), \ell(w)) = \eta_1(v, w)$ for all $v, w \in V_1$; in other words, if $\ell^* \eta_2 = \eta_1$. The set of all symplectic linear isomorphisms from $V, \eta$ to itself is called the symplectic group and denoted $Sp(V, \eta)$. With respect to a symplectic basis $B$ a symplectic linear isomorphism $\ell$ is represented by a matrix $A = [\ell]_B$ that satisfies

$$A^t J A = J$$

where $J = J_n$ is the matrix defined above and where $2n = \dim(V)$. Such a matrix is called a symplectic matrix and the group of all such is called the symplectic matrix group and denoted $Sp(n, \mathbb{R})$. Of course if $\dim(V) = 2n$ then $Sp(V, \eta) \cong Sp(n, \mathbb{R})$ the isomorphism depending a choice of basis. If $\eta$ is a symplectic form on $V$ with $\dim(V) = 2n$ then $\eta^a \in \wedge^{2n} V$ is nonzero and so orients the vector space $V$.

Lemma 10.3 If $A \in Sp(n, \mathbb{R})$ then $\det(A) = 1$.

Proof. If we use $A$ as a linear transformation $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ then $A^* \alpha_0 = \alpha_0$ and $A^* \alpha_0^a = \alpha_0^a$ where $\alpha_0$ is the standard symplectic form on $\mathbb{R}^{2n}$ and $\alpha_0^a \in \wedge^{2n} \mathbb{R}^{2n}$ is top form. Thus $\det(A) = 1$. ■
10.2. CANONICAL FORM (LINEAR CASE)

Theorem 10.4 (Symplectic eigenvalue theorem) If \( \lambda \) is a (complex) eigenvalue of a symplectic matrix \( A \) then so is \( 1/\lambda, \bar{\lambda}, \) and \( 1/\bar{\lambda} \).

Proof. Let \( p(\lambda) = \det(A - \lambda I) \) be the characteristic polynomial. It is easy to see that \( J^t = -J \) and \( JAJ^{-1} = (A^{-1})^t \). Using these facts we have

\[
p(\lambda) = \det(J(A - \lambda I)J^{-1}) = \det(A^{-1} - \lambda I) \\
= \det(A^{-1}(I - \lambda A)) = \det(I - \lambda A) \\
= \lambda^{2n} \det(1/\lambda - A) = \lambda^{2n} p(1/\lambda).
\]

So we have \( p(\lambda) = \lambda^{2n} p(1/\lambda) \). Using this and remembering that 0 is not an eigenvalue one concludes that \( 1/\lambda \) and \( \bar{\lambda} \) are eigenvalues of \( A \). \( \blacksquare \)

Exercise 10.5 With respect to the last theorem, show that \( \lambda \) and \( 1/\lambda \) have the same multiplicity.

10.2 Canonical Form (Linear case)

Suppose one has a vector space \( W \) with dual \( W^* \). We denote the pairing between \( W \) and \( W^* \) by \( \langle ., . \rangle \). There is a simple way to produce a symplectic form on the space \( Z = W \times W^* \) which we will call the canonical symplectic form. This is defined by

\[
\Omega((v_1, \alpha_1), (v_2, \alpha_2)) := \langle \alpha_2, v_1 \rangle - \langle \alpha_1, v_2 \rangle.
\]

If \( W \) is an inner product space with inner product \( \langle ., . \rangle \) then we may form the canonical symplectic from on \( Z = W \times W \) by the same formula. As a special case we get the standard symplectic form on \( \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \) given by

\[
\Omega((x, y), (\tilde{x}, \tilde{y})) = \tilde{y} \cdot x - y \cdot \tilde{x}.
\]

10.3 Symplectic manifolds

We have already defined symplectic manifold in the process of giving example. We now take up the proper study of symplectic manifold. We give the definition again:

Definition 10.6 A symplectic form on a manifold \( M \) is a nondegenerate closed 2-form \( \omega \in \Omega^2(M) = \Gamma(M, T^*M) \). A symplectic manifold is a pair \( (M, \omega) \) where \( \omega \) is a symplectic form on \( M \). If there exists a symplectic form on \( M \) we say that \( M \) has a symplectic structure or admits a symplectic structure.

A map of symplectic manifolds, say \( f : (M, \omega) \to (N, \varpi) \) is called a symplectic map if and only if \( f^* \varpi = \omega \). We will reserve the term symplectomorphism to refer to diffeomorphisms that are symplectic maps. Notice that since a symplectic form such as \( \omega \) is nondegenerate, the \( 2n \) form \( \omega^n = \omega \wedge \cdots \wedge \omega \) is nonzero and global. Hence a symplectic manifold is orientable (more precisely, it is oriented).
Definition 10.7 The form \( \omega = (-1)^n \frac{(2n)!}{n!} \omega^n \) is called the canonical volume form or Liouville volume.

We immediately have that if \( f : (M, \omega) \to (M, \omega) \) is a symplectic diffeomorphism then \( f^*\Omega = \Omega \).

Not every manifold admits a symplectic structure. Of course if \( M \) does admit a symplectic structure then it must have even dimension but there are other more subtle obstructions. For example, the fact that \( H^2(S^4) = 0 \) can be used to show that \( S^4 \) does not admit a symplectic structure. To see this, suppose to the contrary that \( \omega \) is a closed nondegenerate 2-form on \( S^4 \). Then since \( H^2(S^4) = 0 \) there would be a 1-form \( \theta \) with \( d\theta = \omega \). But then since \( d(\omega \wedge \theta) = \omega \wedge \omega \) the 4-form \( \omega \wedge \omega \) would be exact also and Stokes’ theorem would give
\[
\int_{S^4} \omega \wedge \omega = \int_{S^4} d(\omega \wedge \theta) = \int_{\partial S^4} \omega \wedge \theta = 0.
\]
So in fact, \( S^4 \) does not admit a symplectic structure. We will give a more careful examination to the question of obstructions to symplectic structures but let us now list some positive examples.

Example 10.8 (surfaces) Any orientable surface with volume form (area form) qualifies since in this case the volume itself is a closed nondegenerate two form.

Example 10.9 (standard) The form \( \omega = \sum_{i=1}^{n} dx_i \wedge dx_i \) on \( \mathbb{R}^2n \) is the prototypical symplectic form for the theory and makes \( \mathbb{R}^n \) a symplectic manifold. (See Darboux’s theorem 10.22 below)

Example 10.10 (cotangent bundle) We will see in detail below that the cotangent bundle of any smooth manifold has a natural symplectic structure. The symplectic form in a natural bundle chart \((q, p)\) has the form \( \omega = \sum_{i=1}^{n} dq_i \wedge dp_i \).

Example 10.11 (complex submanifolds) The symplectic \( \mathbb{R}^{2n} \) may be considered the realification of \( \mathbb{C}^n \) and then multiplication by \( i \) is thought of as a map \( J : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \). We have that \( \omega_{can}(v, Jv) = -|v|^2 \) so that \( \omega_{can} \) is nondegenerate on any complex submanifold \( M \) of \( \mathbb{R}^{2n} \) and so \( M, \omega_{can}|_M \) is a symplectic manifold.

Example 10.12 (coadjoint orbit) Let \( G \) be a Lie group. Define the coadjoint map \( \text{Ad}^\dagger : G \to \text{GL}(g^*) \), which takes \( g \) to \( \text{Ad}_g^\dagger \), by
\[
\text{Ad}_g^\dagger(\xi)(x) = \xi(\text{Ad}_{g^{-1}}(x)).
\]
The action defined by \( \text{Ad}^\dagger \),
\[
g \to g \cdot \xi = \text{Ad}_g^\dagger(\xi),
\]
is called the coadjoint action. Then we have an induced map \( \text{ad}^\dagger : g \to gl(g^*) \) at the Lie algebra level;
\[
\text{ad}^\dagger(x)(\xi)(y) = -\xi([x, y]).
\]
The orbits of the action given by \( \text{Ad}^\dagger \) are called coadjoint orbits and we will show in theorem below that each orbit is a symplectic manifold in a natural way.
10.4 Complex Structure and Kähler Manifolds

Recall that a complex manifold is a manifold modeled on \( \mathbb{C}^n \) and such that the chart overlap functions are all biholomorphic. Every (real) tangent space \( T_p M \) of a complex manifold \( M \) has a complex structure \( J_p : T_p M \to T_p M \) given in biholomorphic coordinates \( z = x + iy \) by

\[
J_p \left( \frac{\partial}{\partial x^i} \right)_p = \frac{\partial}{\partial y^i} \bigg|_p \\
J_p \left( \frac{\partial}{\partial y^i} \right)_p = - \frac{\partial}{\partial x^i} \bigg|_p
\]

and for any (biholomorphic) overlap function \( \Delta = \varphi \circ \psi^{-1} \) we have \( T \Delta \circ J = J \circ T \Delta \).

**Definition 10.13** An almost complex structure on a smooth manifold \( M \) is a bundle map \( J : TM \to TM \) covering the identity map such that \( J^2 = -\text{id} \).

If one can choose an atlas for \( M \) such that all the coordinate change functions (overlap functions) \( \Delta \) satisfy \( T \Delta \circ J = J \circ T \Delta \) then \( J \) is called a complex structure on \( M \).

**Definition 10.14** An almost symplectic structure on a manifold \( M \) is a nondegenerate smooth 2-form \( \omega \) that is not necessarily closed.

**Theorem 10.15** A smooth manifold \( M \) admits an almost complex structure if and only if it admits an almost symplectic structure.

**Proof.** First suppose that \( M \) has an almost complex structure \( J \) and let \( g \) be any Riemannian metric on \( M \). Define a quadratic form \( q_p \) on each tangent space by

\[
q_p(v) = g_p(v, v) + g_p(Jv, Jv).
\]

Then we have \( q_p(Jv) = q_p(v) \). Now let \( h \) be the metric obtained from the quadratic form \( q \) by polarization. It follows that \( h(v, w) = h(Jv, Jw) \) for all \( v, w \in TM \). Now define a two form \( \omega \) by

\[
\omega(v, w) = h(v, Jw).
\]

This really is skew-symmetric since \( \omega(v, w) = h(v, Jw) = h(Jv, J^2w) = -h(Jv, w) = \omega(w, v) \). Also, \( \omega \) is nondegenerate since if \( v \neq 0 \) then \( \omega(v, Jv) = h(v, Jv) > 0 \).

Conversely, let \( \omega \) be a nondegenerate two form on a manifold \( M \). Once again choose a Riemannian metric \( g \) for \( M \). There must be a vector bundle map \( \Omega : TM \to TM \) such that

\[
\omega(v, w) = g(\Omega v, w) \text{ for all } v, w \in TM.
\]

Since \( \omega \) is nondegenerate the map \( \Omega \) must be invertible. Furthermore, since \( \Omega \) is clearly anti-symmetric with respect to \( g \) the map \( -\Omega \circ \Omega = -\Omega^2 \) must be
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symmetric and positive definite. From linear algebra applied fiberwise we know that there must be a positive symmetric square root for \(-\Omega^2\). Denote this by \(P = \sqrt{-\Omega^2}\). Finite dimensional spectral theory also tell us that \(P\Omega = \Omega P\). Now let \(J = \Omega P^{-1}\) and notice that 

\[
J^2 = (\Omega P^{-1})(\Omega P^{-1}) = \Omega^2 P^{-2} = -\Omega^2 \Omega^{-2} = -\text{id}
\]

One consequence of this result is that there must be characteristic class obstructions to the existence of a symplectic structure on a manifolds. In fact, if \((M, \omega)\) is a symplectic manifold then it is certainly almost symplectic and so there is an almost complex structure \(J\) on \(M\). The tangent bundle is then a complex vector bundle with \(J\) giving the action of multiplication by \(\sqrt{-1}\) on each fiber \(T_p M\). Denote the resulting complex vector bundle by \(TM_J\) and then consider the total Chern class 

\[
c(TM_J) = c_n(TM_J) + ... + c_1(TM_J) + 1.
\]

Here \(c_i(TM_J) \in H^{2i}(M, \mathbb{Z})\). Recall that with the orientation given by \(\omega^n\) the top class \(c_n(TM_J)\) is the Euler class \(e(TM)\) of \(TM\). Now for the real bundle \(TM\) we have the total Pontrijagin class 

\[
p(TM) = p_n(TM) + ... + p_1(TM) + 1
\]

which are related to the Chern classes by the Whitney sum 

\[
p(TM) = c(TM_J) \oplus c(TM^{-J}) = (c_n(TM_J) + ... + c_1(TM_J) + 1)((-1)^n c_n(TM_J) - ... + c_1(TM_J) + 1)
\]

where \(TM^{-J}\) is the complex bundle with \(-J\) giving the multiplication by \(\sqrt{-1}\). We have used the fact that 

\[
c_i(TM^{-J}) = (-1)^i c_i(TM_J).
\]

Now the classes \(p_k(TM)\) are invariants of the diffeomorphism class of \(M\) and so can be considered constant over all possible choices of \(J\). In fact, from the above relations one can deduce a quadratic relation that must be satisfied: 

\[
p_k(TM) = c_k(TM_J)^2 - 2c_{k-1}(TM_J)c_{k+1}(TM_J) + \cdots + (-1)^k 2c_{2k}(TM^{-J}).
\]

Now this places a restriction on what manifolds might have almost complex structures and hence a restriction on having an almost symplectic structure. Of course some manifolds might have an almost symplectic structure but still have no symplectic structure.

**Definition 10.16** A positive definite real bilinear form \(h\) on an almost complex manifold \((M, J)\) is will be called Hermitian metric or \(J\)-metric if \(h\) is \(J\)-invariant. In this case \(h\) is the real part of a Hermitian form on the complex vector bundle \(TM, J\) given by 

\[
(v, w) = h(v, w) + ih(Jv, w)
\]
A diffeomorphism \( \phi : (M, J, h) \to (M, J, h) \) is called a Hermitian isometry if and only if \( T\phi \circ J = J \circ T\phi \) and \( h(T\phi v, T\phi w) = h(v, w) \).

A group action \( \rho : G \times M \to M \) is called a Hermitian action if \( \rho(g, .) \) is a Hermitian isometry for all \( g \). In this case, we have for every \( p \in M \) the representation \( d\rho_p : H_p \to \text{Aut}(T_p M, J_p) \) of the isotropy subgroup \( H_p \) given by \( d\rho_p(g)v = T_p \rho g \cdot v \).

A complex manifold \( M, J \) is called a \textbf{Kähler manifold} if \( h(v, w) := \omega(v, Jw) \) is positive definite.

Equivalently we can define a \textbf{Kähler manifold} as a complex manifold \( M, J \) with Hermitian metric \( h \) with the property that the nondegenerate 2-form \( \omega(v, w) := h(v, Jw) \) is closed.

Thus we have the following for a Kähler manifold:

1. A complex structure \( J \),
2. A \( J \)-invariant positive definite bilinear form \( b \),
3. A Hermitian form \( \langle v, w \rangle = h(v, w) + ih(Jv, w) \).
4. A symplectic form \( \omega \) with the property that \( \omega(v, w) = h(v, Jw) \).

Of course if \( M, J \) is a complex manifold with Hermitian metric \( h \) then \( \omega(v, w) := h(v, Jw) \) automatically gives a nondegenerate 2-form; the question is whether it is closed or not. Mumford’s criterion is useful for this purpose:

\textbf{Theorem 10.19 (Mumford)} Let \( \rho : G \times M \to M \) be a smooth Lie group action by Hermitian isometries. For \( p \in M \) let \( H_p \) be the isotropy subgroup of the point \( p \). If \( J_p \in d\rho_p(H_p) \) for every \( p \) then we have that \( \omega \) defined by \( \omega(v, w) := h(v, Jw) \) is closed.

\textbf{Proof.} It is easy to see that since \( \rho \) preserves both \( h \) and \( J \) it also preserves \( \omega \) and \( d\omega \). Thus for any given \( p \in M \), we have

\[ d\omega(dp_p(g)u, dp_p(g)v, dp_p(g)w) = d\omega(u, v, w) \]

for all \( g \in H_p \) and all \( u, v, w \in T_p M \). By assumption there is a \( g_p \in H_p \) with \( J_p = dp_p(g_p) \). Thus with this choice the previous equation applied twice gives

\[ d\omega(u, v, w) = d\omega(J_p u, J_p v, J_p w) \]
\[ = d\omega(J_p^2 u, J_p^2 v, J_p^2 w) \]
\[ = d\omega(-u, -v, -w) = -d\omega(u, v, w) \]

so \( d\omega = 0 \) at \( p \) which was an arbitrary point so \( d\omega = 0 \).}

Since a Kähler manifold is a posteriori a Riemannian manifold it has associated with it the Levi-Civita connection \( \nabla \). In the following we view \( J \) as an element of \( \mathfrak{X}(M) \).
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Theorem 10.20 For a Kähler manifold \( M, J, h \) with associated symplectic form \( \omega \) we have that
\[ d\omega = 0 \quad \text{if and only if} \quad \nabla J = 0. \]

10.5 Symplectic musical isomorphisms

Since a symplectic form \( \omega \) on a manifold \( M \) is nondegenerate we have a map
\[ \omega^\flat : T^*M \to TM \]
given by \( \omega^\flat(p)(v_p) = \omega(v_p) \) and the inverse \( \omega^\sharp \) is such that
\[ \iota_{\omega^\sharp(\alpha)} \omega = \alpha \]
or
\[ \omega(\omega^\sharp(\alpha_p), v_p) = \alpha_p(v_p) \]
Let check that \( \omega^\sharp \) really is the inverse. (one could easily be off by a sign in this business.) We have
\[ \omega_b(\omega^\sharp(\alpha_p))(v_p) = \omega(\omega^\sharp(\alpha_p), v_p) = \alpha_p(v_p) \quad \text{for all} \quad v_p \]
\[ \implies \omega_b(\omega^\sharp(\alpha_p)) = \alpha_p. \]

Notice that \( \omega^\sharp \) induces a map on sections also denoted by \( \omega^\sharp \) with inverse \( \omega^\flat : \mathcal{X}(M) \to \mathcal{X}^*(M) \).

Notation 10.21 Let us abbreviate \( \omega^\sharp(\alpha) \) to \( \sharp\alpha \) and \( \omega^\flat(v) \) to \( \flat v \).

10.6 Darboux’s Theorem

Lemma 10.22 (Darboux’s theorem) On a \( 2n \)-manifold \( (M, \omega) \) with a closed 2-form \( \omega \) with \( \omega^n \neq 0 \) (for instance if \( (M, \omega) \) is symplectic) there exists a sub-atlas consisting of charts called symplectic charts (canonical coordinates) characterized by the property that the expression for \( \omega \) in such a chart is
\[ \omega_U = \sum_{i=1}^{n} dx^i \wedge dx^{i+n} \]
and so in particular \( M \) must have even dimension \( 2n \).

Remark 10.23 Let us agree that the canonical coordinates can be written \((x^i, y_i)\) instead of \((x^i, x^{i+n})\) when convenient.

Remark 10.24 It should be noticed that if \( x^i, y_i \) is a symplectic chart then \( \sharp dx^i \) must be such that
\[ \sum_{r=1}^{n} dx^r \wedge dy^r(\sharp dx^i, \frac{\partial}{\partial x^j}) = \delta_j^i \]
but also
\[ \sum_{r=1}^{n} dx^r \wedge dy^r(\partial/\partial x^i) = \sum_{r=1}^{n} dx^r(dy^r(\partial/\partial x^i) - dy^r(dx^r(\partial/\partial x^i))) = -dy^i(dx^i) \]
and so we conclude that \( \#dx^i = -\partial/\partial y^i \) and similarly \( \#dy^i = \partial/\partial x^i \).

**Proof.** We will use induction and follow closely the presentation in [?]. Assume the theorem is true for symplectic manifolds of dimension 2\((n-1)\). Let \( p \in M \). Choose a function \( y^1 \) on some open neighborhood of \( p \) such that \( dy_1(p) \neq 0 \). Let \( X = \#dy_1 \) and then \( X \) will not vanish at \( p \). We can then choose another function \( x^1 \) such that \( Xx^1 = 1 \) and we let \( Y = -\#dx^1 \). Now since \( d\omega = 0 \) we can use Cartan’s formula to get
\[ L_X\omega = L_Y\omega = 0. \]

In the following we use the notation \( \langle X, \omega \rangle = \iota_X\omega \) (see notation ??). Contract \( \omega \) with the bracket of \( X \) and \( Y \):
\[ \langle [X, Y], \omega \rangle = \langle \mathcal{L}_X Y, \omega \rangle = \mathcal{L}_X \langle Y, \omega \rangle - \langle Y, \mathcal{L}_X \omega \rangle = \mathcal{L}_X(dx^1) = -d(X(x^1)) = -d1 = 0. \]

Now since \( \omega \) is nondegenerate this implies that \( [X, Y] = 0 \) and so there must be a local coordinate system \( (x^1, y^1, w^1, \ldots, w^{2n-2}) \) with
\[ \frac{\partial}{\partial y^1} = Y \]
\[ \frac{\partial}{\partial x^1} = X. \]
In particular, the theorem is true if \( n = 1 \). Assume the theorem is true for symplectic manifolds of dimension 2\((n-1)\). If we let \( \omega' = \omega - dx^1 \wedge dy_1 \) then since \( d\omega' = 0 \) and hence
\[ \langle X, \omega' \rangle = \mathcal{L}_X \omega' = \langle Y, \omega' \rangle = \mathcal{L}_Y \omega' = 0 \]
we conclude that \( \omega' \) can be expressed as a 2-form in the \( w^1, \ldots, w^{2n-2} \) variables alone. Furthermore,
\[ 0 \neq \omega^n = (\omega - dx^1 \wedge dy_1)^n = \pm ndx^1 \wedge dy_1 \wedge (\omega')^n \]
from which it follows that \( \omega' \) is the pull-back of a form nondegenerate form \( \varpi \) on \( \mathbb{R}^{2n-2} \). To be exact if we let the coordinate chart given by \( (x^1, y^1, w^1, \ldots, w^{2n-2}) \) by denoted by \( \psi \) and let \( pr \) be the projection \( \mathbb{R}^{2n} = \mathbb{R}^2 \times \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1} \) then \( \omega' = (pr \circ \psi)^* \varpi \). Thus the induction hypothesis says that \( \omega' \) has the form
$\omega' = \sum_{i=2}^{n} dx^i \wedge dy_i$ for some functions $x^i, y_i$ with $i = 2, \ldots, n$. It is easy to see that the construction implies that in some neighborhood of $p$ the full set of functions $x^i, y_i$ with $i = 1, \ldots, n$ form the desired symplectic chart. □

An atlas $\mathcal{A}$ of symplectic charts is called a symplectic atlas. A chart $(U, \varphi)$ is called compatible with the symplectic atlas $\mathcal{A}$ if for every $(\psi, U) \in \mathcal{A}$ we have

$$(\varphi \circ \psi^{-1})^* \omega_0 = \omega_0$$

for the canonical symplectic $\omega_{can} = \sum_{i=1}^{n} du^i \wedge du^{i+n}$ defined on $\psi(U \cap U) \subset \mathbb{R}^{2n}$ using standard rectangular coordinates $u^i$.

10.7 Poisson Brackets and Hamiltonian vector fields

**Definition 10.25** (on forms) The Poisson bracket of two 1-forms is defined to be

$$\{\alpha, \beta\} \pm = \pm \omega(\sharp \alpha, \sharp \beta)$$

where the musical symbols refer to the maps $\omega^\sharp$ and $\omega^\flat$. This puts a Lie algebra structure on the space of 1-forms $\Omega^1(M) = X^*(M)$.

**Definition 10.26** (on functions) The Poisson bracket of two smooth functions is defined to be

$$\{f, g\} \pm = \pm \omega(\sharp df, \sharp dg) = \pm \omega(X_f, X_g)$$

This puts a Lie algebra structure on the space $\mathcal{F}(M)$ of smooth function on the symplectic $M$. It is easily seen (using $dg = \iota_{X_f} \omega$) that $\{f, g\} \pm = \pm L_{X_f} f = \mp L_{X_f} g$, which shows that $f \mapsto \{f, g\}$ is a derivation for fixed $g$. The connection between the two Poisson brackets is

$$d\{f, g\} \pm = \{df, dg\} \pm.$$

Let us take canonical coordinates so that $\omega = \sum_{i=1}^{n} dx^i \wedge dy_i$. If $X_p = \sum_{i=1}^{n} dx^i(X) \frac{\partial}{\partial x^i} + \sum_{i=1}^{n} dy_i(X) \frac{\partial}{\partial y_i}$ and $v_p = dx^i(v_p) \frac{\partial}{\partial x^i} + dy_i(v_p) \frac{\partial}{\partial y_i}$, then using the Einstein summation convention we have

$$\omega(Y)(v_p)$$

$$= \omega(dx^i(X) \frac{\partial}{\partial x^i} + dy_i(X) \frac{\partial}{\partial y_i}, dx^i(v_p) \frac{\partial}{\partial x^i} + dy_i(v_p) \frac{\partial}{\partial y_i})$$

$$= (dx^i(X)dy_i - dy_i(X)dx^i)(v_p)$$

so we have

**Lemma 10.27** $\omega(Y)(X_p) = \sum_{i=1}^{n} dx^i(X)dy_i - dy_i(X)dx^i = \sum_{i=1}^{n} (-dy_i(X)dx^i + dx^i(X)dy_i)$
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Corollary 10.28 If \( \alpha = \sum_{i=1}^{n} \alpha_{i} \frac{\partial}{\partial x_{i}} dx_{i} + \sum_{i=1}^{n} \alpha_{i} \frac{\partial}{\partial y_{i}} dy_{i} \) then \( \omega(\alpha) = \sum_{i=1}^{n} \alpha_{i} \frac{\partial}{\partial y_{i}} - \sum_{i=1}^{n} \alpha_{i} \frac{\partial}{\partial x_{i}} \)

An now for the local formula:

Corollary 10.29 \( \{f,g\} = \sum_{i=1}^{n} (\frac{\partial f}{\partial x^{i}} \frac{\partial g}{\partial y^{i}} - \frac{\partial f}{\partial y^{i}} \frac{\partial g}{\partial x^{i}}) \)

Proof. \( df = \frac{\partial f}{\partial x_{i}} dx_{i} + \frac{\partial f}{\partial y_{i}} dy_{i} \) and \( dg = \frac{\partial g}{\partial x_{j}} dx_{j} + \frac{\partial g}{\partial y_{i}} dy_{i} \) so \( \omega(df) = \frac{\partial f}{\partial y_{i}} \frac{\partial}{\partial x_{i}} - \frac{\partial f}{\partial x_{i}} \frac{\partial}{\partial y_{i}} \) and similarly for \( dg \). Thus (using the summation convention again);

\[ \{f,g\} = \omega(\star df, \star dg) \]

\[ = \sum_{i=1}^{n} (\frac{\partial f}{\partial y_{i}} \frac{\partial}{\partial x^{i}} - \frac{\partial f}{\partial x^{i}} \frac{\partial}{\partial y_{i}}) \frac{\partial}{\partial x^{i}} \frac{\partial}{\partial y_{i}} \]

A main point about Poison Brackets is

Theorem 10.30 \( f \) is constant along the orbits of \( X_{g} \) if and only if \( \{f,g\} = 0 \).

In fact, \( \frac{d}{dt} g \circ \varphi_{t}^{X_{g}} = 0 \iff \{f,g\} = 0 \iff \frac{d}{dt} f \circ \varphi_{t}^{X_{g}} = 0 \)

Proof. \( \frac{d}{dt} g \circ \varphi_{t}^{X_{g}} = (\varphi_{t}^{X_{g}})^{*} L_{X_{g}} g = (\varphi_{t}^{X_{g}})^{*} \{f,g\} \). Also use \( \{f,g\} = -\{g,f\} \).

The equations of motion for a Hamiltonian \( H \) are

\[ \frac{d}{dt} f \circ \varphi_{t}^{X_{H}} = \pm \{f \circ \varphi_{t}^{X_{H}}, H\} = \mp \{H, f \circ \varphi_{t}^{X_{H}}\} \]

which is true by the following simple computation

\[ \frac{d}{dt} f \circ \varphi_{t}^{X_{H}} = \frac{d}{dt} (\varphi_{t}^{X_{H}})^{*} f = (\varphi_{t}^{X_{H}})^{*} L_{X_{H}} f \]

\[ = L_{X_{H}} (f \circ \varphi_{t}^{X_{H}}) = \{f \circ \varphi_{t}^{X_{H}}, H\} \]

Notation 10.31 From now on we will use only \( \{.,.\}_{+} \) unless otherwise indicated and shall write \( \{.,.\} \) for \( \{.,.\}_{+} \).

Definition 10.32 A Hamiltonian system is a triple \( (M, \omega, H) \) where \( M \) is a smooth manifold, \( \omega \) is a symplectic form and \( H \) is a smooth function \( H : M \rightarrow \mathbb{R} \).

The main example, at least from the point of view of mechanics, is the cotangent bundle of a manifold which is discussed below. From a mechanical point of view the Hamiltonian function controls the dynamics and so is special.

Let us return to the general case of a symplectic manifold \( M, \omega \).
Definition 10.33 Now if $H : M \to \mathbb{R}$ is smooth then we define the Hamiltonian vector field $X_H$ with energy function $H$ to be $\omega^2 \omega$ so that by definition $i_{X_H} \omega = dH$.

Definition 10.34 A vector field $X$ on $M, \omega$ is called a locally Hamiltonian vector field or a symplectic vector field if and only if $L_X \omega = 0$.

If a symplectic vector field is complete then we have that $(\varphi^X_t)^* \omega$ is defined for all $t \in \mathbb{R}$. Otherwise, for any relatively compact open set $U$ the restriction $\varphi^X_t$ to $U$ is well defined for all $t \leq b(U)$ for some number depending only on $U$. Thus $(\varphi^X_t)^* \omega$ is defined on $U$ for $t \leq b(U)$. Since $U$ can be chosen to contain any point of interest and since $M$ can be covered by relatively compact sets, it will be of little harm to write $(\varphi^X_t)^* \omega$ even in the case that $X$ is not complete.

Lemma 10.35 The following are equivalent:

1. $X$ is symplectic vector field, i.e. $L_X \omega = 0$
2. $i_X \omega$ is closed
3. $(\varphi^X_t)^* \omega = \omega$
4. $X$ is locally a Hamiltonian vector field.

Proof. (1)$\iff$ (4) by the Poincaré lemma. Next, notice that $L_X \omega = d \circ i_X \omega + i_X \circ d \omega = d \circ i_X \omega$ so we have (2)$\iff$ (1). The implication (2)$\iff$ (3) follows from Theorem ??.

Proposition 10.36 We have the following easily deduced facts concerning Hamiltonian vector fields:

1. The $H$ is constant along integral curves of $X_H$
2. The flow of $X_H$ is a local symplectomorphism. That is $\varphi^{X_H}_t \omega = \omega$

Notation 10.37 Denote the set of all Hamiltonian vector fields on $M, \omega$ by $\mathcal{H}(\omega)$ and the set of all symplectic vector fields by $\mathcal{SP}(\omega)$

Proposition 10.38 The set $\mathcal{SP}(\omega)$ is a Lie subalgebra of $\mathfrak{X}(M)$. In fact, we have $[\mathcal{SP}(\omega), \mathcal{SP}(\omega)] \subset \mathcal{H}(\omega) \subset \mathfrak{X}(M)$.

Proof. Let $X, Y \in \mathcal{SP}(\omega)$. Then

$$[X,Y]_\omega = \mathcal{L}_X Y \omega = \mathcal{L}_X (Y \omega) - Y \mathcal{L}_X \omega$$

$$= d(X \omega) + X d(Y \omega) - d \omega$$

$$= d(X \omega) + 0 + 0$$

$$= -d(\omega(X,Y)) = -X_{\omega(X,Y)} \omega$$

and since $\omega$ in nondegenerate we have $[X,Y] = X_{-\omega(X,Y)} \in \mathcal{H}(\omega)$. ■
10.8 Configuration space and Phase space

Consider the cotangent bundle of a manifold \( Q \) with projection map
\[
\pi : T^* Q \to Q
\]
and define the **canonical 1-form** \( \theta \in T^* (T^* Q) \) by
\[
\theta : v_{\alpha_p} \mapsto \alpha_p(T \pi \cdot v_{\alpha_p})
\]
where \( \alpha_p \in T^*_p Q \) and \( v_{\alpha_p} \in T_{\alpha_p}(T^*_p Q) \). In local coordinates this reads
\[
\theta_0 = \sum p_i dq^i.
\]
Then \( \omega_{T^* Q} = -d\theta \) is a symplectic form that in natural coordinates reads
\[
\omega_{T^* Q} = \sum dq^i \wedge dp_i
\]

**Lemma 10.39** \( \theta \) is the unique 1-form such that for any \( \beta \in \Omega^1(Q) \) we have
\[
\beta^* \theta = \beta
\]
where we view \( \beta \) as \( \beta : Q \to T^* Q \).

Proof: \( \beta^* \theta(v_q) = \theta|_{\beta(q)}(T \beta \cdot v_q) = \beta(q)(T \pi \circ T \beta \cdot v_q) = \beta(q)(v_q) \) since \( T \pi \circ T \beta = T(\pi \circ \beta) = T(id) = id \).

The cotangent lift \( T^* f \) of a diffeomorphism \( f : Q_1 \to Q_2 \) is defined by the commutative diagram
\[
\begin{array}{ccc}
T^* Q_1 & \xrightarrow{T^* f} & T^* Q_2 \\
\downarrow & & \downarrow \\
Q_1 & \xrightarrow{f} & Q_2
\end{array}
\]
and is a symplectic map; i.e. \( (T^* f)^* \omega_0 = \omega_0 \). In fact, we even have \( (T^* f)^* \theta_0 = \theta_0 \).

The triple \( (T^* Q, \omega_{T^* Q}, H) \) is a Hamiltonian system for any choice of smooth function. The most common form for \( H \) in this case is \( \frac{1}{2} K + V \) where \( K \) is a Riemannian metric that is constructed using the mass distribution of the bodies modeled by the system and \( V \) is a smooth potential function which, in a conservative system, depends only on \( q \) when viewed in natural cotangent bundle coordinates \( q^i, p_i \).

Now we have \( \partial g = \frac{\partial g}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial g}{\partial q^i} \frac{\partial}{\partial p_i} \) and introducing the \( \pm \) notation one more time we have
\[
\{ f, g \}_\pm = \pm \omega_{T^* Q}(\delta df, \delta dg) = \pm df(\delta dg) = \pm df \left( \frac{\partial g}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial g}{\partial q^i} \frac{\partial}{\partial p_i} \right)
\]
\[
= \pm \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right)
\]
\[
= \pm \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right)
\]
Thus letting
\[ \varphi^X_t(q_0^1, ..., q_0^n, p_0^1, ..., p_0^n) = (q^1(t), ..., q^n(t), p^1(t), ..., p^n(t)) \]
the equations of motions read
\[
\frac{d}{dt}f(q(t), p(t)) = \frac{d}{dt}(f \circ \varphi^X_t) = \{f \circ \varphi^X_t, H\} \\
= \frac{\partial f}{\partial q} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q}.
\]
Where we have abbreviated \( f \circ \varphi^X_t \) to just \( f \). In particular, if \( f = q^i \) and \( f = p_i \) then
\[
\dot{q}^i(t) = \frac{\partial H}{\partial p_i}, \\
\dot{p}_i(t) = -\frac{\partial H}{\partial q}.
\]
which should be familiar.

10.9 Transfer of symplectic structure to the Tangent bundle

Case I: a (pseudo) Riemannian manifold

If \( Q, g \) is a (pseudo) Riemannian manifold then we have a map \( g^\flat : T^*Q \to T^*Q \) defined by
\[ g^\flat(v)(w) = g(v, w) \]
and using this we can define a symplectic form \( \omega_0 \) on \( TQ \) by
\[ \omega_0 = (g^\flat)^* \omega \]
(Note that \( d\omega_0 = d(g^{\flat*}\omega) = g^{\flat*}d\omega = 0. \) In fact, \( \omega_0 \) is exact since \( \omega \) is exact:
\[ \omega_0 = (g^\flat)^* \omega \\
= (g^\flat)^* d\theta = d \left( g^{\flat*} \theta \right). \]
Let us write \( \Theta_0 = g^{\flat*} \theta \). Locally we have
\[ \Theta_0(x, v)(v_1, v_2) = g_x(v, v_1) \text{ or} \\
\Theta_0 = \sum g_{ij} \dot{q}^i dq^j \]
and also
\[ \varpi_0(x,v)(((v_1,v_2),((w_1,w_2))) = g_x(w_2,v_1) - g_x(v_2,w_1) + D_x g_x(v,v_1) \cdot w_1 - D_x g_x(v,w_1) \cdot v \]
which in classical notation (and for finite dimensions) looks like
\[ \varpi_h = g_{ij} dq^i \wedge d\dot{q}^j + \sum \frac{\partial g_{ij}}{\partial q^k} \dot{q}^i dq^j \wedge dq^k \]

Case II: Transfer of symplectic structure by a Lagrangian function.

**Definition 10.40** Let \( L : TQ \to Q \) be a Lagrangian on a manifold \( Q \). We say that \( L \) is **regular** or **non-degenerate** at \( \xi \in TQ \) if in any canonical coordinate system \((q,\dot{q})\) whose domain contains \( \xi \), the matrix
\[ \begin{bmatrix} \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}(q(\xi),\dot{q}(\xi)) \end{bmatrix} \]
is non-degenerate. \( L \) is called **regular** or **nondegenerate** if it is regular at all points in \( TQ \).

We will need the following general concept:

**Definition 10.41** Let \( \pi_E : E \to M \) and \( \pi_F : F \to M \) be two vector bundles. A map \( L : E \to F \) is called a **fiber preserving map** if the following diagram commutes:
\[ E \xrightarrow{L} F \]
\[ \pi_E \quad \pi_F \]
\[ \downarrow \quad \uparrow \]
\[ M \]

We do not require that the map \( L \) be linear on the fibers and so in general \( L \) is not a vector bundle morphism.

**Definition 10.42** If \( L : E \to F \) is a fiber preserving map then if we denote the restriction of \( L \) to a fiber \( E_p \) by \( L_p \), define the **fiber derivative**
\[ FL : E \to \text{Hom}(E,F) \]
by \( FL : e_p \mapsto Df|_p(e_p) \) for \( e_p \in E_p \).

In our application of this concept, we take \( F \) to be the trivial bundle \( Q \times \mathbb{R} \) over \( Q \) so \( \text{Hom}(E,F) = \text{Hom}(E,\mathbb{R}) = T^*Q \).

**Lemma 10.43** A Lagrangian function \( L : TQ \to \mathbb{R} \) gives rise to a fiber derivative \( FL : TQ \to T^*Q \). The Lagrangian is nondegenerate if and only if \( FL \) is a diffeomorphism.
Definition 10.44 The form $\varpi_L$ is defined by
$$\varpi_L = (FL)^*\omega$$

Lemma 10.45 $\omega_L$ is a symplectic form on $TQ$ if and only if $L$ is nondegenerate (i.e. if $F_L$ is a diffeomorphism).

Observe that we can also define $\theta_L = (FL)^*\theta$ so that $d\theta_L = d(FL)^*\theta = (FL)^*d\theta = (FL)^*\omega = \varpi_L$ so we see that $\omega_L$ is exact (and hence closed a required for a symplectic form).

Now in natural coordinates we have
$$\varpi_L = \frac{\partial^2L}{\partial \dot{q}^i \partial \dot{q}^j}dq^i \wedge dq^j + \frac{\partial^2L}{\partial \dot{q}^i \partial \dot{q}^j}d\dot{q}^i \wedge d\dot{q}^j$$
as can be verified using direct calculation.

The following connection between the transferred forms $\varpi_L$ and $\varpi_0$ and occasionally not pointed out in some texts.

Theorem 10.46 Let $V$ be a smooth function on a Riemannian manifold $M, h$. If we define a Lagrangian by $L = \frac{1}{2}h - V$ then the Legendre transformation $F_L : TQ \rightarrow T^*Q$ is just the map $q^i$ and hence $\varpi_L = \varpi_h$.

Proof. We work locally. Then the Legendre transformation is given by
$$q^i \mapsto q^i, \dot{q}^i \mapsto \frac{\partial L}{\partial \dot{q}^i}.$$But since $L(\dot{q}, \dot{q}) = \frac{1}{2}g(\dot{q}, \dot{q}) - V(q)$ we have $\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial}{\partial q^i} \frac{1}{2}g_{kl}\dot{q}^k \dot{q}^l = g_{il}\dot{q}^l$ which together with $q^i \mapsto q^i$ is the coordinate expression for $g^i_l :$
$$\dot{q}^i \mapsto g_{il}q^l.$$10.10 Coadjoint Orbits

Let $G$ be a Lie group and consider $Ad^*: G \rightarrow GL(g^*)$ and the corresponding coadjoint action as in example 10.12. For every $\xi \in g^*$ we have a Left invariant 1-form on $G$ defined by
$$\theta^\xi = \xi \circ \omega_G$$where $\omega_G$ is the canonical $g$-valued 1-form (the Maurer-Cartan form). Let the $G_\xi$ be the isotropy subgroup of $G$ for a point $\xi \in g^*$ under the coadjoint action. Then it is standard that orbit $G : \xi$ is canonically diffeomorphic to the orbit space $G/G_\xi$ and the map $\phi_\xi : g \mapsto g \cdot \xi$ is a submersion onto $\xi$. Then we have
Theorem 10.47  There is a unique symplectic form $\Omega^\xi$ on $G/\xi \cong G \cdot \xi$ such that $\phi^*_\xi \Omega^\xi = d\theta^\xi$.

Proof: If such a form as $\Omega^\xi$ exists as stated then we must have

$$\Omega^\xi(T\phi^*_\xi v, T\phi^*_\xi w) = d\theta^\xi(v, w) \text{ for all } v, w \in T_g G$$

We will show that this in fact defines $\Omega^\xi$ as a symplectic form on the orbit $G \cdot \xi$.

First of all notice that by the structure equations for the Maurer-Cartan form we have for $v, w \in T_e G = g$

$$d\theta^\xi(v, w) = \xi(d\omega_G(v, w)) = \xi(\omega_G([v, w])) = \xi([-v, w]) = \text{ad}^1(v)(\xi)(w)$$

From this we see that

$$\text{ad}^1(v)(\xi) = 0 \iff v \in \text{Null}(d\theta^\xi|_e)$$

where $\text{Null}(d\theta^\xi|_e) = \{v \in g : d\theta^\xi|_e(v, w) \text{ for all } w \in g\}$. On the other hand, $G_\xi = \ker \{g \mapsto \text{Ad}_g^1(\xi)\}$ so $\text{ad}^1(v)(\xi) = 0$ if and only if $v \in T_e G_\xi = g_\xi$.

Now notice that since $d\theta^\xi$ is left invariant we have that $\text{Null}(d\theta^\xi|_g) = TL_g(g_\xi)$ which is the tangent space to the coset $gG_\xi$ and which is also $\ker T\phi^*_\xi|_g$. Thus we conclude that

$$\text{Null}(d\theta^\xi|_g) = \ker T\phi^*_\xi|_g.$$ 

It follows that we have a natural isomorphism

$$T_{g, \xi}(G \cdot \xi) = T\phi^*_\xi|_g(T_g G) \approx T_g G/(TL_g(g_\xi))$$

Another view: Let the vector field on $G \cdot \xi$ corresponding to $v, w \in g$ generated by the action be denoted by $v^\dagger$ and $w^\dagger$. Then we have $\Omega^\xi(\xi|_g(v^\dagger, w^\dagger)) := \xi([-v, w])$ at $\xi \in G \cdot \xi$ and then extend to the rest of the points of the orbit by equivariance:

$$\Omega^\xi(g \cdot \xi|_g(v^\dagger, w^\dagger)) = \text{Ad}_g(\xi)([-v, w])$$

10.11  The Rigid Body

In what follows we will describe the rigid body rotating about one of its points in three different versions. The basic idea is that we can represent the configuration space as a subset of $\mathbb{R}^{3N}$ with a very natural kinetic energy function. But this space is also isomorphic to the rotation group $SO(3)$ and we can transfer the kinetic energy metric over to $SO(3)$ and then the evolution of the system is given by geodesics in $SO(3)$ with respect to this metric. Next we take advantage of the fact that the tangent bundle of $SO(3)$ is trivial to transfer the setup over to a trivial bundle. But there are two natural ways to do this and we explore the relation between the two.
10.11.1 The configuration in \( \mathbb{R}^{3N} \)

Let us consider a rigid body to consist of a set of point masses located in \( \mathbb{R}^3 \) at points with position vectors \( \mathbf{r}_1(t), ..., \mathbf{r}_N(t) \) at time \( t \). Thus \( \mathbf{r}_i = (x_1, x_2, x_3) \) is the coordinates of the \( i \)-th point mass. Let \( m_1, ..., m_N \) denote the masses of the particles. To say that this set of point masses is rigid is to say that the distances \( |\mathbf{r}_i - \mathbf{r}_j| \) are constant for each choice of \( i \) and \( j \). Let us assume for simplicity that the body is in a state of uniform rectilinear motion so that by re-choosing our coordinate axes if necessary we can assume that the there is one of the point masses at the origin of our coordinate system at all times. Now the set of all possible configurations is some submanifold of \( \mathbb{R}^{3N} \) which we denote by \( M \). Let us also assume that at least 3 of the masses, say those located at \( \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_2 \) are situated so that the position vectors \( \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \) form a basis of \( \mathbb{R}^3 \). For convenience let \( \mathbf{r} \) and \( \dot{\mathbf{r}} \) be abbreviations for \( (\mathbf{r}_1(t), ..., \mathbf{r}_N(t)) \) and \( (\dot{\mathbf{r}}_1(t), ..., \dot{\mathbf{r}}_N(t)) \). The correct kinetic energy for the system of particles forming the rigid body is \( \frac{1}{2} K(\dot{\mathbf{r}}, \dot{\mathbf{r}}) \) where the kinetic energy metric \( K \) is

\[
K(\mathbf{v}, \mathbf{w}) = m_1 \mathbf{v}_1 \cdot \mathbf{w}_1 + \cdots + m_N \mathbf{v}_N \cdot \mathbf{w}_N.
\]

Since there are no other forces on the body other than those that constrain the body to be rigid the Lagrangian for \( M \) is just \( \frac{1}{2} K(\dot{\mathbf{r}}, \dot{\mathbf{r}}) \) and the evolution of the point in \( M \) representing the body is a geodesic when we use as Hamiltonian \( K \) and the symplectic form pulled over from \( T^*M \) as described previously.

10.11.2 Modelling the rigid body on \( SO(3) \)

Let \( \mathbf{r}_1(0), ..., \mathbf{r}_N(0) \) denote the initial positions of our point masses. Under these condition there is a unique matrix valued function \( \mathbf{g}(t) \) with values in \( SO(3) \) such that \( \mathbf{r}_i(t) = \mathbf{g}(t)\mathbf{r}_i(0) \). Thus the motion of the body is determined by the curve in \( SO(3) \) given by \( t \mapsto \mathbf{g}(t) \). In fact, we can map \( SO(3) \) to the set of all possible configurations of the points making up the body in a 1-1 manner by letting

\[
\mathbf{r}_1(0) = \xi_1, ..., \mathbf{r}_N(0) = \xi_N
\]

and mapping \( \Phi : \mathbf{g} \mapsto (g\xi_1, ..., g\xi_N) \in M \subset \mathbb{R}^{3N} \). If we use the map \( \Phi \) to transfer this over to \( TSO(3) \) we get

\[
k(\xi, \mathbf{v}) = K(T\Phi \cdot \xi, T\Phi \cdot \mathbf{v})
\]

for \( \xi, \mathbf{v} \in TSO(3) \). Now \( k \) is a Riemannian metric on \( SO(3) \) and in fact, \( k \) is a left invariant metric:

\[
k(\xi, \mathbf{v}) = k(TL_g\xi, TL_g\mathbf{v}) \text{ for all } \xi, \mathbf{v} \in TSO(3).
\]

Exercise 10.48 Show that \( k \) really is left invariant. Hint: Consider the map \( \mu_{g_0} : (\mathbf{v}_1, ..., \mathbf{v}_N) \mapsto (g_0\mathbf{v}_1, ..., g_0\mathbf{v}_N) \) for \( g_0 \in SO(3) \) and notice that \( \mu_{g_0} \circ \Phi = \Phi \circ L_{g_0} \impliedby T \mu_{g_0} \circ T\Phi = T\Phi \circ TL_{g_0} \).

Now by construction, the Riemannian manifolds \( M, K \) and \( SO(3), k \) are isometric. Thus the corresponding path \( g(t) \) in \( SO(3) \) is a geodesic with respect to the left invariant metric \( k \). Our Hamiltonian system is now \( (TSO(3), \Omega_k, k) \) where \( \Omega_k \) is the Legendre transformation of the canonical symplectic form \( \Omega \) on \( T^*SO(3) \).
10.11.3 The trivial bundle picture

Recall that we the Lie algebra of $SO(3)$ is the vector space of skew-symmetric matrices $\mathfrak{su}(3)$. We have the two trivializations of the tangent bundle $T SO(3)$ given by

\[
\text{triv}_L(v_g) = (g, \omega_G(v_g)) = (g, g^{-1}v_g)
\]

\[
\text{triv}_R(v_g) = (g, \omega_G(v_g)) = (g, v_g g^{-1})
\]

with inverse maps $SO(3) \times \mathfrak{so}(3) \to T SO(3)$ given by

\[(g, B) \mapsto T L_g B \]

\[(g, B) \mapsto T R_g B \]

Now we should be able to represent the system in the trivial bundle $SO(3) \times \mathfrak{so}(3)$ via the map $\text{triv}_L(v_g) = (g, \omega_G(v_g)) = (g, g^{-1}v_g)$. Thus we let $k_0$ be the metric on $SO(3) \times \mathfrak{so}(3)$ coming from the metric $k$. Thus by definition

\[k_0((g,v), (g,w)) = k(T L_g v, T L_g w) = k_e(v, w)\]

where $v, w \in \mathfrak{so}(3)$ are skew-symmetric matrices.

10.12 The momentum map and Hamiltonian actions

**Remark 10.49** In this section all Lie groups will be assumed to be connected.

Suppose that (a connected Lie group) $G$ acts on $M, \omega$ as a group of symplectomorphisms.

\[\sigma : G \times M \to M\]

Then we say that $\sigma$ is a symplectic $G$-action. Since $G$ acts on $M$ we have for every $v \in \mathfrak{g}$ the fundamental vector field $X^v = v^\sigma$. The fundamental vector field will be symplectic (locally Hamiltonian). Thus every one-parameter group $g^t$ of $G$ induces a symplectic vector field on $M$. Actually, it is only the infinitesimal action that matters at first so we define

**Definition 10.50** Let $M$ be a smooth manifold and let $\mathfrak{g}$ be the Lie algebra of a connected Lie group $G$. A linear map $\sigma' : v \mapsto X^v$ from $\mathfrak{g}$ into $\mathfrak{X}(M)$ is called a $\mathfrak{g}$-**action** if

\[ [X^v, X^w] = -X^{[v,w]} \text{ or} \]

\[ [\sigma'(v), \sigma'(w)] = -\sigma'([v, w]). \]

If $M, \omega$ is symplectic and the $\mathfrak{g}$-action is such that $\mathcal{L}_{X^v} \omega = 0$ for all $v \in \mathfrak{g}$ we say that the action is a symplectic $\mathfrak{g}$-action.
Definition 10.51 Every symplectic action $\sigma : G \times M \to M$ induces a $\mathfrak{g}$-action $d\sigma$ via

$$d\sigma : v \mapsto X^v$$

where $X^v(x) = \frac{d}{dt} \bigg|_0 \sigma(\exp(tv), x)$.

In some cases, we may be able to show that for all $v$ the symplectic field $X^v$ is a full fledged Hamiltonian vector field. In this case associated to each $v \in \mathfrak{g}$ there is a Hamiltonian function $J_v = J_{X^v}$ with corresponding Hamiltonian vector field equal to $X^v$ and $J_v$ is determined up to a constant by $\iota_{X^v}\omega = \mathcal{L}_{X^v}\omega$. Now is it possible to define $J_v$ for every $v \in \mathfrak{g}$?

Lemma 10.52 Given a symplectic $\mathfrak{g}$-action $\sigma' : v \mapsto X^v$ as above, there is a linear map $v \mapsto J_v$ such that $X^v = \sharp d J_v$ for every $v \in \mathfrak{g}$ if and only if $\iota_{X^v}\omega$ is exact for all $v \in \mathfrak{g}$.

Proof. If $H_v = H_{X^v}$ exists for all $v$ then $dJ_{X^v} = \omega(X^v, \cdot) = \iota_{X^v}\omega$ for all $v$ so $\iota_{X^v}\omega$ is exact for all $v \in \mathfrak{g}$. Conversely, if for every $v \in \mathfrak{g}$ there is a smooth function $h_v$ with $dh_v = \iota_{X^v}\omega$ then $X^v = \sharp dh_v$ so $h_v$ is Hamiltonian for $X^v$. Now let $v_1, \ldots, v_n$ be a basis for $\mathfrak{g}$ and define $J_v = h_{v_i}$ and extend linearly. \[\blacksquare\]

Notice that the property that $v \mapsto J_v$ is linear means that we can define a map $J : M \to \mathfrak{g}^*$ by $J(x)(v) = J_v(x)$ and this is called a momentum map.

Definition 10.53 A symplectic $G$-action $\sigma$ (resp. $\mathfrak{g}$-action $\sigma'$) on $M$ such that for every $v \in \mathfrak{g}$ the vector field $X^v$ is a Hamiltonian vector field on $M$ is called a Hamiltonian $G$-action (resp. Hamiltonian $\mathfrak{g}$-action).

We can thus associate to every Hamiltonian action at least one momentum map this being unique up to an additive constant.

Example 10.54 If $G$ acts on a manifold $Q$ by diffeomorphisms then $G$ lifts to an action on the cotangent bundle $T^*M$ which is automatically symplectic. In fact, because $\omega_0 = d\theta_0$ is exact the action is also a Hamiltonian action. The Hamiltonian function associated to an element $v \in \mathfrak{g}$ is given by $J_v(x) = \theta_0 \left( \frac{d}{dt} \bigg|_0 \exp(tv) \cdot x \right)$.

Definition 10.55 If $G$ (resp. $\mathfrak{g}$) acts on $M$ in a symplectic manner as above such that the action is Hamiltonian and such that we may choose a momentum map $J$ such that $J_{[v,w]} = \{J_v, J_w\}$ where $J_v(x) = J(x)(v)$ then we say that the action is a strongly Hamiltonian $G$-action (resp. $\mathfrak{g}$-action).
Example 10.56 The action of example 10.54 is strongly Hamiltonian.

We would like to have a way to measure of whether a Hamiltonian action is strong or not. Essentially we are just going to be using the difference $J_{[\upsilon, w]} = \{ J_\upsilon, J_w \}$ but it will be convenient to introduce another view which we postpone until the next section where we study “Poisson structures”.

PUT IN THEOREM ABOUT MOMENTUM CONSERVATION!!!!

What is a momentum map in the cotangent case? Pick a fixed point $\alpha \in T^*Q$ and consider the map $\Phi_\alpha : G \rightarrow T^*Q$ given by $\Phi_\alpha(g) = g \cdot \alpha = g^{-1}\ast \alpha$. Now consider the pull-back of the canonical 1-form $\Phi_\alpha^*\theta_0$.

Lemma 10.57 The restriction $\Phi_\alpha^*\theta_0|_g$ is an element of $\mathfrak{g}^*$ and the map $\alpha \mapsto \Phi_\alpha^*\theta_0|_g$ is the momentum map.

Proof. We must show that $\Phi_\alpha^*\theta_0|_g(v) = H_\upsilon(\alpha)$ for all $v \in \mathfrak{g}$. Does $\Phi_\alpha^*\theta_0|_g(v)$ live in the right place? Let $g_v = \exp(\upsilon t)$. Then

$$ (T_e \Phi_\alpha) \upsilon = \frac{d}{dt} \bigg|_0 \Phi_\alpha(\exp(\upsilon t)) = \frac{d}{dt} \bigg|_0 (\exp(-\upsilon t))^* \alpha = \frac{d}{dt} \bigg|_0 \exp(\upsilon t) \cdot \alpha $$

We have

$$ \Phi_\alpha^*\theta_0|_g(v) = \theta_0|_g (T_e \Phi_\alpha) \upsilon = \theta_0(\frac{d}{dt} \bigg|_0 \exp(\upsilon t) \cdot \alpha) = J_\upsilon(\alpha) $$

Definition 10.58 Let $G$ act on a symplectic manifold $M, \omega$ and suppose that the action is Hamiltonian. A momentum map $J$ for the action is said to be equivariant with respect to the coadjoint action if $J(g \cdot x) = \text{Ad}_{g^{-1}}^* J(x)$. 


Chapter 11

Poisson Geometry

Life is good for only two things, discovering mathematics and teaching mathematics
—Siméon Poisson

11.1 Poisson Manifolds

In this chapter we generalize our study of symplectic geometry by approaching things from the side of a Poisson bracket.

Definition 11.1 A Poisson structure on an associative algebra $A$ is a Lie algebra structure with bracket denoted by $\{,\}$ such for a fixed $a \in A$ that the map $x \mapsto \{a, x\}$ is a derivation of the algebra. An associative algebra with a Poisson structure is called a Poisson algebra and the bracket is called a Poisson bracket.

We have already seen an example of a Poisson structure on the algebra $\mathfrak{F}(M)$ of smooth functions on a symplectic manifold. Namely,

$$\{f, g\} = \omega(\omega^\sharp df, \omega^\sharp dg).$$

By the Darboux theorem we know that we can choose local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ on a neighborhood of any given point in the manifold. Recall also that in such coordinates we have

$$\omega^\sharp df = \sum_{i=1}^n \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \sum_{i=1}^n \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i}$$

sometimes called the symplectic gradient. It follows that

$$\sum_{i=1}^n \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right)$$
A smooth manifold with a Poisson structure on is algebra of smooth functions is called a **Poisson manifold**.

So every symplectic $n$-manifold gives rise to a Poisson structure. On the other hand, there are Poisson manifolds that are not so by virtue of being a symplectic manifold.

Now if our manifold is finite dimensional then every derivation of $\mathfrak{g}(M)$ is given by a vector field and since $g \mapsto \{f, g\}$ is a derivation there is a corresponding vector field $X_f$. Since the bracket is determined by these vector field and since vector fields can be defined locally (recall the presheaf $\mathcal{X}_M$) we can see that a Poisson structure is also a locally defined structure. In fact, $U \mapsto \mathfrak{g}_M(U)$ is a presheaf of Poisson algebras.

Now if we consider the map $w : \mathfrak{g}_M \to \mathcal{X}_M$ defined by $\{f, g\} = w(f) \cdot g$ we see that $\{f, g\} = w(f) \cdot g = -w(g) \cdot f$ and so $\{f, g\}(p)$ depends only on the differentials $df, dg$ of $f$ and $g$. Thus we have a tensor $B(\cdot, \cdot) \in \Gamma \wedge^2 TM$ such that $B(df, dg) = \{f, g\}$. In other words, $B_p(\cdot, \cdot)$ is a symmetric bilinear map $T_p^* M \times T_p^* M \to \mathbb{R}$. Now any such tensor gives a bundle map $B^2 : T^* M \to T^* M$ by the rule $B^2(\alpha)(\beta) = B(\beta, \alpha)$ for $\beta, \alpha \in T_p^* M$ and any $p \in M$.

In other words, $B(\beta, \alpha) = \beta(B^2(\alpha))$ for all $\beta \in T_p^* M$ and arbitrary $p \in M$. The 2-vector $B$ is called the Poisson tensor for the given Poisson structure. $B$ is also sometimes called a cosymplectic **structure** for reasons that we will now explain.

If $M, \omega$ is a symplectic manifold then the map $\omega : TM \to T^* M$ can be inverted to give a map $\omega^\sharp : T^* M \to TM$ and then a form $W \in \bigwedge^2 TM$ defined by $\omega^\sharp(\alpha)(\beta) = W(\beta, \alpha)$ (here again $\beta, \alpha$ must be in the same fiber). Now this form can be used to define a Poisson bracket by setting $\{f, g\} = W(df, dg)$ and so $W$ is the corresponding Poisson tensor. But notice that

\[
\{f, g\} = W(df, dg) = \omega^\sharp(dg)(df) = df(\omega^\sharp(dg)) = \omega(\omega^\sharp df, \omega^\sharp dg)
\]

which is just the original Poisson bracket defined in the symplectic manifold $M, \omega$.

Given a Poisson manifold $M, \{\cdot, \cdot\}$ we can always define $\{\cdot, \cdot\}_-$ by $\{f, g\}_- = \{g, f\}$. Since we some times refer to a Poisson manifold $M, \{\cdot, \cdot\}$ by referring just to the space we will denote $M$ with the opposite Poisson structure by $M^\circ$.

A Poisson map is map $\phi : M, \{\cdot, \cdot\}_1 \to N, \{\cdot, \cdot\}_2$ is a smooth map such that $\phi^* \{f, g\} = \{\phi^* f, \phi^* g\}$ for all $f, g \in \mathfrak{g}(M)$.

For any subset $S$ of a Poisson manifold let $S_0$ be the set of functions from $\mathfrak{g}(M)$ that vanish on $S$. A submanifold $S$ of a Poisson manifold $M, \{\cdot, \cdot\}_1$ is called **coisotropic** if $S_0$ closed under the Poisson bracket. A Poisson manifold is called symplectic if the Poisson tensor $B$ is non-degenerate since in this case we can use $B^\sharp$ to define a symplectic form on $M$. A Poisson manifold admits a (singular) foliation such that the leaves are symplectic. By a theorem of A. Weinstein we can locally in a neighborhood of a point $p$ find a coordinate system...
11.1. POISSON MANIFOLDS

\((q^i, p_i, w^i)\) centered at \(p\) and such that

\[
B = \sum_{i=1}^{k} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i,j} a^{ij} \left( \frac{\partial}{\partial w^i} \wedge \frac{\partial}{\partial w^j} \right)
\]

where the smooth functions depend only on the \(w\)'s. vanish at \(p\). Here \(k\) is the dimension of the leaf through \(p\). The rank of the map \(B^\#\) on \(T^*_p M\) is \(k\).

Now to give a typical example let \(g\) be a Lie algebra with bracket \([., .]\) and \(g^\ast\) its dual. Choose a basis \(e_1, ..., e_n\) of \(g\) and the corresponding dual basis \(\epsilon^1, ..., \epsilon^n\) for \(g^\ast\). With respect to the basis \(e_1, ..., e_n\) we have

\[
[e_i, e_j] = \sum C_{ij}^k e_k
\]

where \(C_{ij}^k\) are the structure constants.

For any functions \(f, g \in g^\ast\) we have that \(df_\alpha, dg_\alpha\) are linear maps \(g^\ast \to \mathbb{R}\) where we identify \(T_\alpha g^\ast\) with \(g^\ast\). This means that \(df_\alpha, dg_\alpha\) can be considered to be in \(g\) by the identification \(g^{**} = g\). Now define the \(\pm\) Poisson structure on \(g^\ast\) by

\[
\{f, g\}_\pm(\alpha) = \pm \alpha([df_\alpha, dg_\alpha])
\]

Now the basis \(e_1, ..., e_n\) is a coordinate system \(y\) on \(g^\ast\) by \(y_i(\alpha) = \alpha(e_i)\).

**Proposition 11.3** In terms of this coordinate system the Poisson bracket just defined is

\[
\{f, g\}_\pm = \pm \sum_{i=1}^{n} B_{ij} \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial y_j}
\]

where \(B_{ij} = \sum C_{ij}^k y_k\).

**Proof.** We suppress the \(\pm\) and compute:

\[
\{f, g\} = [df, dg] = \left[ \sum \frac{\partial f}{\partial y_i} dy_i, \sum \frac{\partial g}{\partial y_j} dy_j \right]
\]

\[
= \sum \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial y_j} [dy_i, dy_j] = \sum \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial y_j} \sum C_{ij}^k y_k
\]

\[
= \sum_{i=1}^{n} B_{ij} \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial y_j}
\]